IMS

MATHS

BOOK-02

Set - I

PARTIAL DIFFERENTIAL EQUATIONS

Partial diff. eqn: An eqn involving the derivatives of a dependent variable w.r.t more than one independent variable, is called a PDE.

Ex: (1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz^2$

(2) $\frac{\partial^2 z}{\partial x^2} = k\left(\frac{\partial^3 z}{\partial x^3}\right)^2$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Order of PDE: The order of the highest order derivative involving in a differential eqn is called the order of the PDE.

The examples (1), (2) and (3) orders are one, three & two respectively.

Degree of PDE: The degree (i.e, power) of the highest order derivative involving in the diff. eqn is called the degree of PDE.

The above examples (1), (2) & (3) degrees are one, two and one.

Linear partial diff. eqn: A partial diff eqn is said to be linear if (i) the dependent variable say z and all its partial derivatives occur in first degree only and (ii) no product of dependent variable (or) partial derivatives occur.

Ex: (1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$

(2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

} are linear.

(3) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz^2$

(4) $\frac{\partial^2 z}{\partial x^2} = k\left(\frac{\partial^3 z}{\partial x^3}\right)^2$

} are not linear.

→ An eqn which is not linear is called non-linear PDE.

→ In the case of two independent variables x and y will usually be taken as the independent and z as the dependent variable.

→ The partial diff. coefficients $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are denoted by p & q.

i.e, $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$.

→ The second order partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are denoted by r, s, t.

i.e, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, and $t = \frac{\partial^2 z}{\partial y^2}$

NOTE: [1] In the case of n independent variables, we take them to be $x_1, x_2, \ldots x_n$ and z as the dependent variable. In this case we use the following notations.

$P_1 = \frac{\partial z}{\partial x_1}$, $P_2 = \frac{\partial z}{\partial x_2}$, $P_3 = \frac{\partial z}{\partial x_3} \ldots\ldots P_n = \frac{\partial z}{\partial x_n}$.

[2]. Some times the partial derivatives are also denoted by suffixes.

$u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$; $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$ and so on.

Formation (Derivation) of PDE:

– partial diff. eqns can derived in two ways.

(I) By the elimination of arbitrary constants from a relation b/w x, y and z

and (II) By the elimination of arbitrary functions of three variables.

[I]. By the elimination of arbitrary constants:

Let z be a function of x and y such that

$f(x, y, z, a, b) = 0$ —— (1) where a & b are arbitrary constants

Differentiating (1) partially w.r.t x & y we get,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0$$

i.e., $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0$ —(2) and $\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0$ —(3)

Now eliminating 'a' and 'b' from (2) & (3) we obtain an eqn of the form

$$f(x, y, z, p, q) = 0 \quad \text{—(4)}$$

which is the required PDE of first order.

Note: If the number of arbitrary constants to be eliminated is equal to the number of independent variables then the derived partial differential eqn is of the first order.

But if the number of arbitrary constants to be eliminated is greater than number of independent variables then the derived partial diff. eqns will be of the second order (or) higher orders.

II. By the elimination of arbitrary functions.

Suppose we have a relation between x, y and z of the type $f(u, v) = 0$ —(1)

where u and v are known as functions of x, y & z and f is arbitrary function of u & v.

Now we treat z dependent variable and x & y are independent variables.

Differentiating (1) w.r.t x we get,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + 0 + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + 0 + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \quad \left(\because p=\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow \frac{\partial f}{\partial u}\Big/\frac{\partial f}{\partial v} = -\frac{\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)}{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)} \quad \text{---(2)}$$

Similarly differentiating (1) w.r.t y we get

$$\frac{\partial f}{\partial u}\Big/\frac{\partial f}{\partial v} = -\frac{\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)}{\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)} \quad \text{---(3)} \quad \left(\because q=\frac{\partial z}{\partial y}\right)$$

Now eliminating f from (2) & (3) we get

$$\frac{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p}{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p} = \frac{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q}{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q}$$

$$\Rightarrow \left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)=\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$\Rightarrow P\cdot p+Q\cdot q=R \quad \text{---(4)}$$

where $P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}=\frac{\partial(u,v)}{\partial(y,z)}$

$Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u,v)}{\partial(z,x)}$

$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(x,y)}$

the eqn (4) is a PDE of the first order.

Note: [1]. If the given relation between x, y, z contains two arbitrary functions then the derived partial diff. eqn will contain partial derivatives of an order higher than two except in particular cases.

[2]. The PDE (4) derived in [II] is a linear i.e, powers of p & q are both unity while the PDE (4) derived in [I] need not be linear.

Type-I

(1) → Form a PDE by elimination of arbitrary constants a & b from the eqn $z = ax + by + a^2 + b^2$.

Solⁿ: Given eqn is $z = ax + by + a^2 + b^2$ ——— (1)

Diff. (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = a \quad \text{——— (2)}$$

$$\frac{\partial z}{\partial y} = b \quad \text{——— (3)}$$

Now eliminating a, b from (1), (2) & (3) we get

$$z = \frac{\partial z}{\partial x} x + \frac{\partial z}{\partial y} y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

which is the required PDE.

(2) → Eliminate a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$.

Solⁿ: Given eqn is $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ ——— (1)

Diff (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = ae^y \quad \text{——— (2)}$$

$$\frac{\partial z}{\partial y} = axe^y + a^2e^{2y}$$

$$= x(ae^y) + (ae^y)^2 \quad \text{——— (3)}$$

Now sub (2) in eqn (3)

$$(3) \Rightarrow \frac{\partial z}{\partial y} = x\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2$$

which is the required PDE.

→ Form a PDE by eliminating arbitrary constants from the following relations.

(3) $z = ax + (1-a)y + b$; a, b

(4) $az + b = a^2x + y$; a, b

(5) $z = (x-a)^2 + (y-b)^2$; a, b

(6) $z = a(x+y) + b$; a, b

(7) $z = ax + by + ab$; a, b

(8) $z = ax + a^2y + b$; a, b

(9) $z = (x+a)(y+b)$; a, b

(10) $z = Ae^{pt}\sin px$; p, t.

→ Form a PDE by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol^n: Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

Differentiating (1) w.r.t x & y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}=0 \Rightarrow \frac{x}{a^2}+\frac{z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{—— (2)}$$

and $$\frac{2y}{b^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial y}=0 \Rightarrow \frac{y}{b^2}+\frac{z}{c^2}\frac{\partial z}{\partial y}=0 \quad \text{—— (3)}$$

Differentiating (2) w.r.t x and (3) w.r.t y, we find

$$\frac{1}{a^2}+\frac{1}{c^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{z}{c^2}\frac{\partial^2 z}{\partial x^2}=0 \Rightarrow c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \quad \text{—— (4)}$$

and $$\frac{1}{b^2}+\frac{1}{c^2}\left(\frac{\partial z}{\partial y}\right)^2+\frac{z}{c^2}\frac{\partial^2 z}{\partial y^2}=0 \Rightarrow c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2+b^2z\frac{\partial^2 z}{\partial y^2}=0 \quad \text{—— (5)}$$

from (2) $$c^2=-\frac{a^2z}{x}\frac{\partial z}{\partial x} \quad \text{—— (6)}$$

Sub (6) in (4), we get

$$(4) \equiv -\frac{a^2z}{x}\frac{\partial z}{\partial x}+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0$$

$$\Rightarrow a^2\left[-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}\right]=0$$

(or) $$-z\frac{\partial z}{\partial x}+x\left(\frac{\partial z}{\partial x}\right)^2+zx\frac{\partial^2 z}{\partial x^2}=0$$

$$\Rightarrow xz\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0 \quad \text{—— (7)}$$

Similarly, from (3) & (5),

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \quad \text{—— (8)}$$

Eqns (7) & (8) are two possible forms of the required equations of order 2.

IAS-96

Q(1) Find the differential eqn of all spheres of radius λ, having centre in the xy-plane. (-I.A.S-96)

Sol: The eqn of any sphere of radius λ, having centre $(h, k, 0)$ in the xy-plane is given by

$(x-h)^2 + (y-k)^2 + (z-0)^2 = \lambda^2$ where h and k are arbitrary constants

$\Rightarrow (x-h)^2 + (y-k)^2 + z^2 = \lambda^2$ ——(1)

Differentiating eqn (1) partially w.r.t x & y resp

$(x-h) + z\frac{\partial z}{\partial x} = 0 \Rightarrow (x-h) = -zp$ ——(2) $\left(\because \frac{\partial z}{\partial x} = p\right)$

$(y-k) + z\frac{\partial z}{\partial y} = 0 \Rightarrow (y-k) = -zq$ ——(3) $\left(\because \frac{\partial z}{\partial y} = q\right)$

Sub (2) & (3) in eqn (1)

$z^2p^2 + z^2q^2 + z^2 = \lambda^2$

$z^2(p^2+q^2+1) = \lambda^2$.

which is the required partial differential equation

IAS-97

(2) Form the differential eqn by eliminating a and b from $z = (x^2+a)(y^2+b)$

Sol: Given $z = (x^2+a)(y^2+b)$. ——(1)

Differentiating (1) partially w.r.t x & y, we get

$\frac{\partial z}{\partial x} = 2x(y^2+b)$

$p = 2x(y^2+b)$

$\Rightarrow y^2+b = p/2x$ ——(2)

$\frac{\partial z}{\partial y} = 2y(x^2+a)$

$q = 2y(x^2+a)$ ——(3)

$\Rightarrow x^2+a = q/2y$

Sub (2) and (3) in eq (1)

$z = \frac{p}{2x} \cdot \frac{q}{2y}$

$\Rightarrow z = \frac{pq}{4xy} \Rightarrow 4xyz = pq$.

which is the required partial differential equation

(3) IAS-98 Find the differential equation of the set of all right circular cones whose axes coincide with z-axis.

Sol: The general equation of the set of all right circular cones whose axes coincide with z-axis having semivertical angle α and vertex at (0,0,c) is given by

$$x^2+y^2 = (z-c)^2 \tan^2\alpha \quad \text{--- (1)}$$

where α and c are arbitrary constants.

Differentiating eq (1) partially w.r.t x & y,

$$2x = 2(z-c)\tan^2\alpha \cdot \frac{\partial z}{\partial x}$$

$$\Rightarrow x = p(z-c)\tan^2\alpha \quad \text{--- (2)}$$

$$2y = 2(z-c)\frac{\partial z}{\partial y}\tan^2\alpha$$

$$\Rightarrow y = (z-c)q\tan^2\alpha \quad \text{--- (3)}$$

from (3) $z-c = \dfrac{y}{q\tan^2\alpha}$ --- (4)

Sub (4) in eq (2)

$$x = \frac{py}{q\tan^2\alpha}\tan^2\alpha$$

$$x = py/q$$

$\Rightarrow qx = py$ which is the required equation.

(4) IAS-98 Eliminate a, b and c from $z = a(x+y) + b(x-y) + abt + c$

Given $z = a(x+y) + b(x-y) + abt + c$ --- (1)

Differentiating eq (1) partially w.r.t x, y & t

$$\frac{\partial z}{\partial x} = a+b \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = a - b \quad \text{——— (3)}$$

$$\frac{\partial z}{\partial t} = ab \quad \text{——— (4)}$$

w.k.t. $4ab = (a+b)^2 - (a-b)^2$

from (2), (3) & (4)

$$4\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

(5) IAS 79, 2002

Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that $yz + zx + xy = 0$ is a surface satisfying the above equation.

Soln: The eqn of any cone with vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{——— (1)}$$

where a, b, c, f, g, h are parameters.

Differentiating eq(1) partially w.r.t. x & y:

$$2ax + 2cz\frac{\partial z}{\partial x} + 2fy\frac{\partial z}{\partial x} + 2g\left(x\frac{\partial z}{\partial x} + z\right) + 2hy = 0$$

$$ax + gz + hy + p(cz + fy + gx) = 0 \quad \text{——— (2)}$$

$$2by + 2cz\frac{\partial z}{\partial y} + 2f\left(y\frac{\partial z}{\partial y} + z\right) + 2hx + 2gx\frac{\partial z}{\partial y} = 0$$

$$by + fz + hx + q(cz + fy + gx) = 0 \quad \text{——— (3)}$$

Multiplying (2) by x and (3) by y and adding, we have

$$ax^2 + gzx + hxy + p(czx + fyx + gx^2) + by^2 + fzy + hxy + q(czy + fy^2 + gxy) = 0$$

$$\Rightarrow (ax^2 + by^2 + gzx + fyz + 2hxy) + px(cz + fy + gx) + qy(cz + fy + gx) = 0$$

$$\Rightarrow (ax^2 + by^2 + gzx + fyz + 2hxy) + (px + qy)(cz + fy + gx) = 0 \quad \text{——— (4)}$$

from eqn (1) $ax^2 + by^2 + 2hxy + gzx + fyz = -cz^2 - fyz - gzx$ (5)

Sub eqn (5) in eqn (4)

$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0$

$\Rightarrow -z(cz + fy + gx) + (cz + fy + gx)(px + qy) = 0$

$\Rightarrow (cz + fy + gx)(px + qy - z) = 0$

$\Rightarrow px + qy - z = 0$ —— (A)

which is the required differential equation.

Given surface is $yz + zx + xy = 0$ —— (6)

Differentiating eq (6) partially w.r.t x and y.

$y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}x + z + y = 0$ and $y\frac{\partial z}{\partial y} + z + x\frac{\partial z}{\partial y} + x = 0$

$\Rightarrow yp + px + z + y = 0$ $\Rightarrow yq + z + xq + x = 0$

$\Rightarrow p(x+y) + z + y = 0$ $\Rightarrow (x+y)q + (x+z) = 0$

$\Rightarrow p = -\frac{(z+y)}{x+y}$ —— (7) $\Rightarrow q = -\frac{(x+z)}{x+y}$ —— (8)

sub (7) and (8) in eqn (A)

$px + qy - z = \frac{-(z+y)x}{x+y} + \frac{-(x+z)y}{x+y} - z$

$= \frac{-xz - xy - xy - zy - xz - yz}{x+y}$

$= \frac{-2xz - 2xy - 2yz}{x+y}$

$= \frac{-2(xz + yz + xy)}{x+y}$

$= 0$ [$\because xy + yz + zx = 0$ by eqn (6)]

$\therefore$ eqn (6) is a surface satisfying eqn (A)

Type-II

(1) Form a PDE by eliminating the arbitrary function ϕ from $z = e^{ny}\phi(x-y)$. —— (1)

Sol^n: Differentiating (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = e^{ny}\phi'(x-y)$$

$$\Rightarrow p = e^{ny}\phi'(x-y) \quad —— (2) \quad \left(\because \frac{\partial z}{\partial x} = p\right)$$

and $q = ne^{ny}\phi(x-y) + e^{ny}\phi'(x-y)(-1)$

$$q = ne^{ny}\phi(x-y) - e^{ny}\phi'(x-y) \quad —— (3)$$

Sub (1) & (2) in eq^n (3)

$$q = nz - p$$

$$\Rightarrow \boxed{p + q = nz}$$

which is the required PDE of order one.

(2) Form a PDE by eliminating the arbitrary functions f and F from $z = f(x+ay) + F(x-ay)$

Sol^n Given $z = f(x+ay) + F(x-ay)$ —— (1)

Diff. (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = f'(x+ay) + F'(x-ay) \quad —— (2)$$

and $\frac{\partial z}{\partial y} = af'(x+ay) - aF'(x-ay)$ —— (3)

Diff. (2) & (3) partially w.r.t x & y respectively, we get.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + F''(x-ay) \quad —— (4)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 F''(x-ay).$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2[f''(x+ay) + F''(x-ay)] \quad —— (5)$$

now sub (4) in (5)

we get $\boxed{\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}}$ which is the required PDE

2007 (3) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$; f is an arbitrary function.

Sol: Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— ①

Diff ① partially w.r.t x & y, we get

$$p = 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(-\frac{1}{x^2}\right) \text{ —— ②}$$

and $q = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y}$ —— ③

② ⇒ $2f'\left(\frac{1}{x} + \log y\right) = -px^2$ —— ④

Sub ④ in ③, we get

$$q = 2y - px^2 \cdot \frac{1}{y}$$

$$\Rightarrow qy = 2y^2 - px^2$$

$$\Rightarrow \boxed{px^2 + qy = 2y^2}$$

which is the required PDE of order one

(4) $z = x^n f(y/x)$; f is an arbitrary function

Sol: Given $z = x^n f(y/x)$ —— ①

Diff ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = n x^{n-1} f(y/x) + x^n f'(y/x) \cdot \left(-y/x^2\right) \text{ —— ②}$$

and $\frac{\partial z}{\partial y} = x^n f'(y/x) \cdot \left(\frac{1}{x}\right)$

$$\Rightarrow x\frac{\partial z}{\partial y} = x^n f'(y/x) \text{ —— ③}$$

② ⇒ $\frac{\partial z}{\partial x} = n \cdot \frac{x^n}{x} f(y/x) + x^n f'(y/x) \cdot \left(-\frac{y}{x^2}\right)$ —— ④

Sub ① & ③ in ④, we get

$$\frac{\partial z}{\partial x} = \frac{n}{x} z + x\left(\frac{\partial z}{\partial y}\right)\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{nz}{x} - \frac{y}{x}\left(\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$ which is the required PDE

→ form partial diff. equs by eliminating the arbitrary functions from the following equations.

(5) $z = f(x+iy) + F(x-iy)$

(6) $z = e^{ax+by} f(ax-by)$

(7) $lx+my+nz = \phi(x^2+y^2+z^2)$

(8) $z = f(x^2-y^2)$

(9) $z = f(x^2+y^2)$

(10) $z = f(y/x)$

(11) → $\phi(x+y+z, x^2+y^2-z^2) = 0$.

Sol: Given $\phi(x+y+z, x^2+y^2-z^2) = 0$.

Let $u = x+y+z$, $v = x^2+y^2-z^2$

Then the given equ. is $\phi(u, v) = 0$ —(1)

Diff (1) w.r.t x partially, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\partial\phi}{\partial u}(1+p) + \frac{\partial\phi}{\partial v}(2x-2zp) = 0 \quad \text{—(2)}$$

$$\left(\because \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial z} = 1; \frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial z} = -2z\right)$$

Diff (1) w.r.t y partially, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\partial\phi}{\partial u}(1+q) + \frac{\partial\phi}{\partial v}(2y-2qz) = 0 \quad \text{—(3)}$$

from (2) $\dfrac{\partial\phi}{\partial u} \Big/ \dfrac{\partial\phi}{\partial v} = \dfrac{-2(x-zp)}{1+p}$ —(4)

and from (3) $\dfrac{\partial\phi}{\partial u} \Big/ \dfrac{\partial\phi}{\partial v} = \dfrac{-2(y-qz)}{1+q}$ —(5)

From (4) & (5)

$$\frac{-2(x-zp)}{(1+p)} = \frac{-2(y-qz)}{(1+q)}$$

$\Rightarrow (1+q)(x-zp) = (1+p)(y-qz)$

$\Rightarrow x+qx-zp-zpq = y+py-qz-pqz$

$\Rightarrow \boxed{p(y+z)-(x+z)q = x-y}$

which is the required PDE of order one.

Q(2) 1996

$z = f(x^2-y) + g(x^2+y)$

Sol: Given $z = f(x^2-y) + g(x^2+y)$ —— (1)

Diff (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = f'(x^2-y)\,2x + g'(x^2+y)\,2x.$$

$$= 2x\left[f'(x^2-y) + g'(x^2+y)\right] \quad —— (2)$$

and $$\frac{\partial z}{\partial y} = f'(x^2-y)(-1) + g'(x^2+y) \quad —— (3)$$

Diff (2) & (3) partially w.r.t x & y respectively

$$\frac{\partial^2 z}{\partial x^2} = 2x\left[f''(x^2-y)(2x) + g''(x^2+y)\cdot 2x\right] + 2\left[f'(x^2-y) + g'(x^2+y)\right]$$

$$= 4x^2\left[f''(x^2-y) + g''(x^2+y)\right] + 2\left[f'(x^2-y) + g'(x^2+y)\right] \quad —— (4)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y) \quad —— (5)$$

from eqn (2) $f'(x^2-y) + g'(x^2+y) = \frac{1}{2x}\left(\frac{\partial z}{\partial x}\right)$ —— (6)

Sub (5) & (6) in (4)

$$\frac{\partial^2 z}{\partial x^2} = 4x^2\left(\frac{\partial^2 z}{\partial y^2}\right) + 2\cdot\left(\frac{1}{2x}\right)\left(\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow \boxed{x\frac{\partial^2 z}{\partial x^2} = 4x^3\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x}}$$

which is the required P.D.E.

Equations solvable by direct integrations

we now consider the PDE's which can be solved by direct integration. In place of the usual constants of integration, we must use arbitrary functions of the variable held fixed.

(1) solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

Sol:

Integrating twice w.r.t x. and keeping y fixed, we get

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x-y) = f(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x-y) = xf(y) + g(y).$$

Now integrating w.r.t y & keeping x fixed, we get

$$z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\int f(y)\,dy + \int g(y)\,dy + w(x)$$

Taking $\int f(y)\,dy = u(y)$

$\int g(y)\,dy = v(y)$

$$z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\,u(y) + v(y) + w(x)$$

where u, v, w are arbitrary functions.

(2) solve $\frac{\partial^2 z}{\partial x^2} + z = 0$; given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$

Ans: $z = \sin x + e^y \cos x$

→ solve the following eqns:

(3) $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

(4) $\frac{\partial^2 z}{\partial x^2} = xy$

(5) $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$

(6) $\frac{\partial^2 z}{\partial x^2} = a^2 z$; given that when $x = 0$, $\frac{\partial z}{\partial x} = a\sin x$ and $\frac{\partial z}{\partial y} = 0$

PDE of order one:-

Classification of first order partial diff. eqns are: (1) linear, (2) semi-linear (3) quasilinear and (4) non-linear eqns.

(1) Linear eqn: A first-order eqn $f(x,y,z,p,q)=0$ is known as linear if it is linear in p, q and z.

i.e, if the given eqn is of the form

$$P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$$

Ex: (1) $yx^2 p + xy^2 q = xyz + x^2y^3$

(2) $p+q = z + xy$.

(2) Semi-linear: A first order partial diff. eqn $f(x,y,z,p,q)=0$ is known as semi-linear eqn if it is linear in p and q and the coefficients of p & q are functions x & y only.

i.e, if the given eqn is of the form

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

Ex: (1) $xyp + x^2yq = x^2y^2z^2$

(2) $yp + xq = \frac{x^2z^2}{y^2}$.

(3) Quasi-linear eqns: A first order PDE $f(x,y,z,p,q)=0$ is known as quasi-linear eqn, if it is linear p & q.

i.e, if the given eqn of the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

Ex (1) $x^2zp + y^2zq = xy$.

(2) $(x^2-yz)p + (y^2-zx)q = z^2 - xy$.

(4) Non-linear: A first order PDE $f(x,y,z,p,q)=0$ which doesnot come under above three types, is known as a

non-linear eq^n.

Ex: (1) $p^2+q^2=1$

(2) $pq=z$

(3) $x^2p^2+y^2q^2=z^2$.

Def: A linear PDE of the first order is known as Lagrange's linear eq^n, is of the form $Pp+Qq=R$ —(1)
where P, Q, R are functions of x, y, z.
This eq^n is called a quasi-linear equation.
This eq^n (1) is obtained by eliminating an arbitrary function f from $f(u,v)=0$ —(2)
where u, v are functions of x, y, z.

Theorem: The general solution of the linear PDE
1987 $Pp+Qq=R$ —(1) is $f(u,v)=0$ —(2)
where f is arbitrary function
and $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ form a solution —(3)
of the equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ —(4)
where P, Q, R are functions of x, y, z.

Proof:
Now diff. (2) partially w.r.t. x & y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

and $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$

Now eliminating $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q \end{vmatrix}=0$$

$$\Rightarrow\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q$$
$$=\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$

$\Rightarrow Pp + Qq = R$

where $P = \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)$

$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$

$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

Which is of the same form of eqn (1)

$\therefore$ (3) is g.s. of (1).

Now consider $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$

where c_1 & c_2 are arbitrary constants.

By differentiating, we get

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \text{---(i)}$$

and $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0$

$$\Rightarrow \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad \text{---(ii)}$$

By cross multiplication we get

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which is same as the eqn (4)

$\therefore u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ are solutions of (4).

Note: Equations (4) are called Lagrange's auxiliary eqns (or) subsidiary eqns for (1).

→ Working Rule for solving of Lagrange's eqn $Pp + Qq = R$:

Step 1: Write the given eqn in standard form $Pp + Qq = R$ ---(1)

Step 2: Write the Lagrange's auxiliary eqns for (1). namely $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ---(2)

Step 3: Solve these simultaneous eqns ② by using the well known methods.

Let $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ be two independent solutions of ②.

Step 4: Write the g.s. of ① as $f(u,v) = 0$ (or) $u = \phi(v)$ (or) $v = \phi(u)$

Methods to solve the simultaneous eqns $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$:

Given eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ —— ①

where P, Q, R are functions of x, y, z

It can be solved in three methods.

Consider the three sets of eqns.

$$\frac{dx}{P} = \frac{dy}{Q} \; ; \; \frac{dx}{P} = \frac{dz}{R} \; ; \; \frac{dy}{Q} = \frac{dz}{R} \quad \text{—— ②}$$

Method [1]: If any two eqns of ② are integrable by the method of variables seperable, we find their general solutions and that pair of solutions form the complete solution of the system ①.

Method [2]: If one eqn of ② only integrable, by the method of variables seperable, we can find its g.s and this solution may be used to find the solution of another set of eqn ②. The pair of these solutions give the g.s of the given equation ①.

Method [3]: ~~If no eqn of ② is integrable then we~~

write $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R} = \frac{l_2 dx + m_2 dy + n_2 dz}{l_2 P + m_2 Q + n_2 R}$

where l_1, m_1, n_1 ; l_2, m_2, n_2 are real numbers or functions of x, y, z

Case(i) If we choose l_1, m_1, n_1 and l_2, m_2, n_2 such that $l_1P + m_1Q + n_1R = 0$ and $l_2P + m_2Q + n_2R = 0$ then $l_1dx + m_1dy + n_1dz = 0$ and $l_2dx + m_2dy + n_2dz = 0$ which on integration gives two eqns.

∴ These eqns together give the complete solution.

Case(ii): If we choose l_1, m_1, n_1 & l_2, m_2, n_2 such that $l_1P + m_1Q + n_1R \neq 0$; $\frac{l_1dx + m_1dy + n_1dz}{l_1P + m_1Q + n_1R} = d\phi$

and $l_2P + m_2Q + n_2R \neq 0$; $\frac{l_2dx + m_2dy + n_2dz}{l_2P + m_2Q + n_2R} = d\psi$

then $\phi(x, y, z) = c_1$, $\psi(x, y, z) = c_2$ will become the g.s. of system (1).

Note: l_1, m_1, n_1 and l_2, m_2, n_2 are called multipliers.

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Problems based on Method [I]

(1) Solve $x^2p + y^2q = z^2$.

Sol^n: Given $x^2p + y^2q = z^2$ —— ①

Clearly which is in the form of $Pp + Qq = R$

Here $P = x^2$; $Q = y^2$; $R = z^2$

Now the Lagrange's auxiliary eqns of ① are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \quad \text{—— ②}$$

Now taking the first two fractions of ②, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \boxed{-\frac{1}{x} + \frac{1}{y} = c_1} \text{—— (3)}$$

Now taking the first and the last fractions of ②,

we get $\frac{dx}{x^2} = \frac{dz}{z^2} \Rightarrow \boxed{-\frac{1}{x} + \frac{1}{z} = c_2}$ —— (4)

∴ From (3) & (4) the required g.s. of ① is

$$f\left(-\frac{1}{x} + \frac{1}{y},\ -\frac{1}{x} + \frac{1}{z}\right) = 0$$

where f is an arbitrary function.

(2) Solve $\left(\frac{y^2z}{x}\right)p + xzq = y^2$

Sol^n: Given that $\left(\frac{y^2z}{x}\right)p + xzq = y^2$ —— ①

Clearly which is in the form of $Pp + Qq = R$

Here $P = \frac{y^2z}{x}$; $Q = xz$ and $R = y^2$.

Now the Lagrange's auxiliary eqns of ① are

$$\frac{dx}{y^2z/x} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{—— ②}$$

Taking the first two fractions of ②, we get

$$\frac{x\,dx}{y^2z} = \frac{dy}{xz} \Rightarrow \frac{x\,dx}{y^2} = \frac{dy}{x}$$

$$\Rightarrow x^2dx = y^2dy$$

$$\Rightarrow x^3 - y^3 = c_1 \quad \text{—— (3)}$$

Taking the first and last fractions of ②, we get

$$\frac{x\,dx}{y^2z} = \frac{dz}{y^2} \Rightarrow x\,dx = z\,dz$$

$$\Rightarrow x^2 - z^2 = 2c_2 \quad \text{—— (4)}$$

$\therefore$ The g.s. of ① is $f(x^3-y^3, x^2-z^2)=0$

where f is arbitrary function.

(3) Solve $a(p+q)=z$

(4) Solve $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\sin x$

(5) Solve $zp=-x$.

(6) Solve $p\tan x+q\tan y=\tan z$.

(7) Solve $y^2p-xyq=x(z-2y)$

Sol^n: Given that $y^2p-xyq=x(z-2y)$ —— ①

which is in the form of $Pp+Qq=R$

$P=y^2$; $Q=-xy$, $R=x(z-2y)$.

Now the Lagrange's auxiliary eqns of ① are

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)} \quad \text{——} ②$$

Taking the first two fractions of ②

$$\frac{dx}{y^2}=\frac{dy}{-xy} \Rightarrow -xdx=ydy$$

$$\Rightarrow \boxed{x^2+y^2=c_1} \quad \text{——} ③$$

Taking the last two fractions of ②, we get

$$\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dz}{dy}=\frac{2y-z}{y}$$

$$\Rightarrow \frac{dz}{dy}+\left(\frac{1}{y}\right)z=2 \quad \text{——} ④$$

$$I.F=e^{\int\frac{1}{y}dy}=e^{\log y}=y$$

G.S. of ④ is

$$z\cdot y=\int 2y\,dy+c_2$$

$$zy=y^2+c_2$$

$$\Rightarrow \boxed{zy-y^2=c_2} \quad \text{——} ④$$

$\therefore$ The required g.s. of ① is $f(x^2+y^2, zy-y^2)=0$

where f is an arbitrary function

Problems based on Method [2]:

(1) Solve $p + 3q = 5z + \tan(y - 3x)$.

Sol: Given that $p + 3q = 5z + \tan(y-3x)$ ——— (1)

Comparing (1) with $Pp + Qq = R$

$P = 1,\ Q = 3,\ R = 5z + \tan(y-3x)$

Now the Lagrange's A. Eqns of (1) are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)} \quad \text{——— (2)}$$

Now taking first two fractions of (2), we get

$$\frac{dx}{1} = \frac{dy}{3} \Rightarrow \frac{dy}{3} = dx \Rightarrow dy = 3dx$$

$$\Rightarrow \boxed{y - 3x = c_1} \quad \text{——— (3)}$$

Now taking last two fractions of (2), we get

$$\frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

$$\Rightarrow \frac{dy}{3} = \frac{dz}{5z+\tan c_1} \quad (\text{from (3)})$$

$$\Rightarrow \frac{1}{3}y = \frac{1}{5}\log(5z + \tan c_1) + c_2$$

$$\Rightarrow \frac{1}{3}y - \frac{1}{5}\log\left[5z + \tan c_1\right] = c_2$$

$$\Rightarrow \frac{1}{3}y - \frac{1}{5}\log\left[5z + \tan(y-3x)\right] = c_2 \quad \text{——— (4)}$$

$\therefore$ G.S. of (1) is

$$f\left(y-3x,\ \frac{1}{3}y - \frac{1}{5}\log(5z + \tan(y-3x))\right) = 0$$

where f is an arbitrary function.

(2) Solve $z(z^2+xy)(px-qy) = x^4$.

Sol: Given that $z(z^2+xy)(px-qy) = x^4$

$$\Rightarrow z(z^2+xy)px - z(z^2+xy)qy = x^4$$

$$\Rightarrow xz(z^2+xy)p + \left[-yz(z^2+xy)\right]q = x^4 \quad \text{——— (1)}$$

Comparing (1) with $Pp + Qq = R$

$P = xz(z^2+xy)$; $Q = -yz(z^2+xy)$.

Now Lagrange's A.E's of (1) are

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4} \quad \text{--- (2)}$$

Taking first two fractions of (2), we get

$$\boxed{xy = c_1} \quad \text{--- (3)}$$

Taking first and last fractions of (2), we get

$$\frac{dx}{zx(z^2+xy)} = \frac{dz}{x^4}$$

$$\Rightarrow \frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4} \quad (\text{from (3)})$$

$$\Rightarrow \frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3} \Rightarrow x^3\,dx = (z^3+c_1 z)\,dz$$

$$\Rightarrow \frac{x^4}{4} = \frac{z^4}{4} + \frac{c_1 z^2}{2} + c_2$$

$$\Rightarrow \boxed{x^4 - z^4 - 2z^2(xy) = 4c_2} \quad \text{--- (4)}$$

$\therefore$ G.S of (1) is

$$f(xy,\ x^4 - z^4 - 2z^2xy) = 0$$

where f is an arbitrary function.

(3) Solve $xzp + yzq = xy$

(4) Solve $p - 2q = 3x^2 \sin(y+2x)$

Problems based on Method (3): Case(i)

(1) Solve $(mz-ny)p + (nx-lz)q = ly-mx$

Sol^n Given that $(mz-ny)p + (nx-lz)q = ly-mx$ --- (1)

Comparing (1) with $Pp + Qq = R$

$P = mz-ny$; $Q = nx-lz$; $R = ly-mx$

Now the Lagrange's A.E of (1) are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad \text{--- (2)}$$

Now using the multipliers $x, y \& z$

each fraction of ② $= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$\boxed{x^2 + y^2 + z^2 = 2c_1}$ — ③

Again using the multipliers l, m, n

each fraction of ② $= \dfrac{l\,dx + m\,dy + n\,dz}{0}$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow \boxed{lx + my + nz = c_2}$ — ④

$\therefore$ from ③ & ④, the required g.s. of ① is

$f(u, v) = 0$

(2) $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Multipliers are x, y, z & $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

(3) $x(y - z)p + y(z - x)q = z(x - y)$

Multipliers are $1, 1, 1$ & $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

(4) $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$

multipliers are x, y, z & $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$

(5) $(y^2 + z^2)p - xyq = -zx$. Multipliers: x, y, z & (no need for the multipliers by method 1)

2004 (6) $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

(7) Solve $(x - y)p + (x + y)q = 2xz$ — ①

Sol^n: $\dfrac{dx}{x - y} = \dfrac{dy}{x + y} = \dfrac{dz}{2xz}$ — ②

Taking first two fractions of ②, we get

$\dfrac{dx}{x - y} = \dfrac{dy}{x + y} \Rightarrow (x + y)\,dx + (y - x)\,dy = 0$

$\Rightarrow (x\,dx + y\,dy) + (y\,dx - x\,dy) = 0$

$\Rightarrow \dfrac{x\,dx + y\,dy}{x^2 + y^2} + \dfrac{y\,dx - x\,dy}{x^2 + y^2} = 0$

$\frac{1}{2} d \log(x^2+y^2) + d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$

$$\boxed{\frac{1}{2}\log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) = c_1} \quad \text{——③}$$

using the multipliers $1, 1, -\frac{1}{z}$

each fraction of ② $= \frac{dx+dy-\frac{1}{z}dz}{(x-y)+(x+y)-\frac{1}{z}(2xz)}$

$= \frac{dx+dy-\frac{1}{z}dz}{2x-2x}$

$\Rightarrow \frac{dx+dy-\frac{1}{z}dz}{0}$

$$\Rightarrow \boxed{x+y-\log z = c_2} \quad \text{——④}$$

∴ from ③ & ④ the required g.s. of ① is

$f(u,v) = 0.$

Case(ii)

1997 → solve $(y+z)p + (z+x)q = x+y$ ——①

Sol: $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ ——②

Using the multipliers $1, -1, 0$

each fraction of ① $= \frac{dx-dy}{y-x}$

$= \frac{d(x-y)}{-(x-y)}$ ——③

Again using the multipliers $0, 1, -1$.

each fraction of ① $= \frac{dy-dz}{z-y}$

$= \frac{dy-dz}{-(y-z)}$ ——④

Finally using multipliers $1, 1, 1$.

each fraction of ① $= \frac{dx+dy+dz}{2(x+y+z)}$

$= \frac{d(x+y+z)}{2(x+y+z)}$ ——⑤

from (3), (4) & (5), we have

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{——⑥}$$

Taking first two fractions of (6)

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{(y-z)}$$

$\Rightarrow \log(x-y) = \log(y-z) + \log c_1$

$\Rightarrow \boxed{\frac{x-y}{y-z} = c_1}$ —— (7)

Now Taking the last two fractions of (6), we get

$\log(x+y+z) + \log(y-z)^2 = \log c_2$

$\Rightarrow \boxed{(x+y+z)(y-z)^2 = c_2}$ —— (8)

$\therefore$ from (7) & (8),

the required g.s. of (1) is

$$f\left(\frac{x-y}{y-z},\ (y-z)^2(x+y+z)\right) = 0$$

1996 Solve $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$ —— (1)

Sol^n: $\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)}$ —— (2)

from (2) $\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)}$

$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-x^2}$

$\Rightarrow x^2dx + y^2dy = 0$

$\Rightarrow \boxed{x^3 + y^3 = 3c_1}$ —— (3)

Choosing the multipliers 1, −1, 0; we get

each fraction of (2) $= \frac{dx-dy}{(x-y)(x^2+y^2)}$ —— (4)

Now equating third fraction of (2) & the fraction (4)

we get, $\frac{dz}{z(x^2+y^2)} = \frac{d(x-y)}{(x-y)(x^2+y^2)}$

$\log z = \log(x-y) + \log c_2$

$\Rightarrow \boxed{\frac{z}{x-y} = c_2}$ —— (5)

$\therefore$ from (3) & (5) the required g.s. of (1) is $f(u,v) = 0$

→ Solve $(x^2-y^2-z^2)p + 2xy\,q = 2xz$. multipliers; x, y, z

→ $(1+y)p + (1+x)q = z$; multipliers; $1, 1, 0$

→ $xz\,p + yz\,q = xy$; multipliers; $\frac{1}{x}, \frac{1}{y}, 0$

→ Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$.

Sol: Given that $(x^2-yz)p + (y^2-zx)q = z^2-xy$ —— ①

Comparing with $Pp + Qq = R$.

$P = x^2-yz$; $Q = y^2-zx$; $R = z^2-xy$

Now the Lagrange's A.E's are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad \text{——} ②$$

Now using the multipliers $1, -1, 0$ and $0, 1, -1$

we get, each fraction of ② =

$$\frac{dx-dy}{x^2-y^2-yz+zx} = \frac{dy-dz}{y^2-z^2-zx+xy}$$

$$\Rightarrow \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{y^2-z^2+x(y-z)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$\Rightarrow \frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} \quad \text{on integration}$$

$$\Rightarrow \boxed{\frac{x-y}{y-z} = c_1} \quad \text{——} ③$$

Using the multipliers $1, 1, 1$; we get

each fraction of ② $= \dfrac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$ —— ④

Again using the multipliers x, y, z, we get

each from fraction of ② $= \dfrac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz}$

$= \dfrac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$ —— ⑤

from (4) & (5) we have

$$\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} = \frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$\Rightarrow (x+y+z)(dx+dy+dz) = x\,dx + y\,dy + z\,dz$

$\Rightarrow \frac{(x+y+z)^2}{2} = \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}+C_2$

$\Rightarrow (x+y+z)^2-(x^2+y^2+z^2) = 2C_2$

$\Rightarrow \boxed{xy+yz+zx = C_2}$ —— (6)

$\therefore$ From (3) & (6), the required g.s of (1) is $f\left(\frac{x-y}{y-z},\ xy+yz+zx\right)=0$ where f is an arbitrary function.

2) Solve $\cos(x+y)\,p+\sin(x+y)\,q = z$ —— (1)

Soln: $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$ —— (2)

Now using the multipliers 1,1,0 and 1,−1,0. each fraction of (2) $= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$ —— (3)

From (2) & (3) we have

$$\frac{dz}{z} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)} \quad —— (4)$$

Now taking last two fractions of (4)

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

$$\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}\,d(x+y) = d(x-y)$$

$\Rightarrow \log[\cos(x+y)+\sin(x+y)] = (x-y)+\log c_1$

$\Rightarrow \boxed{[\cos(x+y)+\sin(x+y)]\,e^{y-x} = c_1}$ —— (5)

Now taking first two fractions of (4), we get

$$\frac{dz}{z} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)}$$

$$\Rightarrow \frac{dz}{z} = \frac{\frac{1}{\sqrt{2}}(dx+dy)}{\sin\left(x+y+\frac{\pi}{4}\right)}$$

$$\Rightarrow \frac{dz}{z} = \frac{1}{\sqrt{2}} \operatorname{cosec}\left(x+y+\frac{\pi}{4}\right) d(x+y)$$

$$\Rightarrow \sqrt{2}\log z = \log\left|\tan\left(\frac{x+y+\frac{\pi}{4}}{2}\right)\right| + \log c_2$$

$$\Rightarrow \log z^{\sqrt{2}} = \log \tan\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right) + \log c_2$$

$$\Rightarrow z^{\sqrt{2}} \cot\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right) = c_2 \quad \text{—— (6)}$$

$\therefore$ From (5) & (6)
the required g.s. of (1) is

$$f\left\{\left[\cos(x+y)+\sin(x+y)\right]e^{y-x},\ z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right)\right\} = 0$$

where f is an arbitrary function.

Q3 → Solve $(x^3+3xy^2)p + (y^3+3x^2y)q = 2z(x^2+y^2)$

multipliers $1, +1, 0$ & $1, -1, 0$

$\frac{1}{x}, \frac{1}{y}, 0$

Q → $p+q = x+y+z$

multipliers $: 1, 1, 1$

Q5 → $(2x^2+y^2+z^2-2yz-zx-xy)p + (x^2+xy^2+z^2-yz-2zx-xy)q$
$= x^2+y^2+2z^2-yz-zx-2xy$

multipliers : $1,-1,0;\ 0,1,-1;\ -1,0,1$

* The Linear eqn containing more than two independent variables:

The generalisation of Lagrange's method is as follows:

Let the linear eqn with 'n' independent variables $x_1, x_2, \ldots, x_n$ be

$$P_1 p_1 + P_2 p_2 + \ldots P_n p_n = R \quad \text{(1)}$$

where $P_1, P_2, \ldots P_n$ and R are fns of $x_1, x_2, \ldots x_n$ and z

Here p_i denotes $\frac{\partial z}{\partial x_i}$; $i = 1, 2, \ldots n$.

Then g.s of (1) is given by

$$f(u_1, u_2, \ldots u_n) = 0$$

where $u_i(x_1, x_2, \ldots x_n, z) = c_i$, $i = 1, 2, \ldots n$, are independent solns of the auxiliary eqns

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

solve

→ $x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 + x_1 x_2 x_3 = 0$

Given eqn is $x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 = -x_1 x_2 x_3$ — (1)

Comparing (1) with $P_1 p_1 + P_2 p_2 + P_3 p_3 = R$.

$\therefore P_1 = x_2 x_3,\ P_2 = x_3 x_1,\ P_3 = x_1 x_2$ & $R = -x_1 x_2 x_3$

Now the Lagrange's auxiliary eqns of (1) are

$$\frac{dx_1}{x_2 x_3} = \frac{dx_2}{x_3 x_1} = \frac{dx_3}{x_1 x_2} = \frac{dz}{-x_1 x_2 x_3} \quad \text{(2)}$$

Taking first two fractions of (2),

$$\frac{dx_1}{x_2 x_3} = \frac{dx_2}{x_3 x_1} \Rightarrow \frac{dx_1}{x_2} = \frac{dx_2}{x_1} \Rightarrow x_1 dx_1 = x_2 dx_2$$

$$\Rightarrow \boxed{x_1^2 - x_2^2 = c_1} \quad \text{(3)}$$

2000 → P.T if $x_1^3 + x_2^3 + x_3^3 = 1$ when $z = 0$, the soln of the eqn $(s-x_1)p_1 + (s-x_2)p_2 + (s-x_3)p_3 = s-z$ can be given in the form

$$s^3\left\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right\}^4=(x_1+x_2+x_3-3z)^3$$

where $s = x_1+x_2+x_3+z$ and $p_i = \frac{\partial z}{\partial x_i}$, $i=1,2,3$

Soln. Given that

$(s-x_1)p_1 + (s-x_2)p_2 + (s-x_3)p_3 = s-z$ —— (1)

where $s = x_1+x_2+x_3+z$ —— (2)

∴ The Lagrange's AE's of (1) are

$$\frac{dx_1}{s-x_1} = \frac{dx_2}{s-x_2} = \frac{dx_3}{s-x_3} = \frac{dz}{s-z}$$

$$\Rightarrow \frac{dx_1}{x_2+x_3+z} = \frac{dx_2}{x_3+x_1+z} = \frac{dx_3}{x_1+x_2+z} = \frac{dz}{x_1+x_2+x_3} \quad \text{—— (3)} \quad \text{(by using (2))}$$

each fraction of (3) is equal to

$$= \frac{dx_1+dx_2+dx_3-3dz}{2(x_1+x_2+x_3)+3z-3(x_1+x_2+x_3)}$$

$$= \frac{dx_1+dx_2+dx_3-3dz}{-(x_1+x_2+x_3)+3z}$$

$$= \frac{dx_1+dx_2+dx_3-3dz}{-(x_1+x_2+x_3-3z)} = \frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} \quad \text{—— (4)}$$

Again, each fraction of (3) $= \frac{dx_1+dx_2+dx_3+dz}{3(x_1+x_2+x_3+z)}$

$$= \frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \quad \text{—— (5)}$$

from (4) and (5)

$$\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} = \frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \Rightarrow \frac{d(x_1+x_2+x_3+z)}{x_1+x_2+x_3+z} + \frac{3d(x_1+x_2+x_3-3z)}{x_1+x_2+x_3-3z} = 0$$

integrating

$$\log(x_1+x_2+x_3+z) + 3\log(x_1+x_2+x_3-3z) = \log a$$

$$\Rightarrow (x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = a \quad \text{--- (6)}$$

where a is any arbitrary constant.

Given That $x_1^3+x_2^3+x_3^3 = 1$ when $z=0$

∴ Eqn (6) gives $(x_1+x_2+x_3)^3(x_1+x_2+x_3) = a$

$$\Rightarrow a = (x_1+x_2+x_3)^4 \quad \text{--- (7)}$$

from (6) & (7)

$$\boxed{(x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = (x_1+x_2+x_3)^4} \quad \text{--- (8)}$$

Now each fraction of (3) $= \dfrac{dx_1+dz}{-(x_1-z)}$

$$= \frac{3(x_1-z)^2\, d(x_1-z)}{-3(x_1-z)^3}$$

$$= \frac{d(x_1-z)^3}{-3(x_1-z)^3} \quad \text{--- (9)}$$

By symmetry, each fraction of (3) is also

$$= \frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3} \quad \text{--- (10)}$$

Using (9) and (10)

each fraction of (3) :

$$= \frac{d(x_1-z)^3}{-3(x_1-z)^3} = \frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3}$$

$$= \frac{d\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]}{-3\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]} \quad \text{--- (11)}$$

from (ii) and (11), we have

$$\frac{3\,d(x_1+x_2+x_3-3z)}{(x_1+x_2+x_3-3z)} = \frac{d\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]}{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3}$$

Integrating

$$3\log(x_1+x_2+x_3-3z)+\log b = \log\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]$$

$$(x_1-z)^3+(x_2-z)^3+(x_3-z)^3 = b(x_1+x_2+x_3-3z)^3 \quad \text{--- (12)}$$

where b is an arbitrary constant.

Given that $x_1^3+x_2^3+x_3^3 = 1$ when $z=0$.

from eqn (12)

$$x_1^3+x_2^3+x_3^3 = b(x_1+x_2+x_3)^3.$$

$$\Rightarrow 1 = b(x_1+x_2+x_3)^3$$

$$\Rightarrow \boxed{b = \frac{1}{(x_1+x_2+x_3)^3}}$$

$\therefore$ (12) $\Rightarrow$

$$(x_1-z)^3+(x_2-z)^3+(x_3-z)^3 = \frac{(x_1+x_2+x_3-3z)^3}{(x_1+x_2+x_3)^3}$$

Raising both sides to power 4, we get

$$\Rightarrow \left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]^4 = \frac{(x_1+x_2+x_3-3z)^{12}}{(x_1+x_2+x_3)^{12}} \quad \text{--- (13)}$$

Raising both sides of eqn (8) to power 3, we have

$$(x_1+x_2+x_3+z)^3(x_1+x_2+x_3-3z)^9 = (x_1+x_2+x_3)^{12} \quad \text{--- (14)}$$

multiplying the corresponding sides of (13) and (14), we have.

$$\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]^4 (x_1+x_2+x_3+z)^3 = (x_1+x_2+x_3-3z)^3$$

$$s^3\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]^4 = (x_1+x_2+x_3-3z)^3$$

Since $x_1+x_2+x_3+z = s$

Integral surfaces passing through a given curve

To find the integral surface of the general solution of the linear partial differential eqn $Pp+Qq=R$ which passes through a given curve.

Let $Pp+Qq=R$ —① be the given eqn.

Let its auxiliary eqns give the following two independent solutions.

$u(x,y,z)=c_1$ & $v(x,y,z)=c_2$ ——②

then g.s. of ① is $f(u,v)=0$ where f is arbitrary function arising from a relation $f(c_1,c_2)=0$ ——③ between the constants c_1 & c_2.

we have to consider the problem of determining the function f in special cases.

Method(1): If we want to find integral surface passing through the given curve whose eqn in parametric form is given by $x=x(t)$, $y=y(t)$, $z=z(t)$, where t is parameter.

then ② may be expressed as

$u(x(t), y(t), z(t))=c_1$ and $v(x(t), y(t), z(t))=c_2$ ——④

Now eliminating the parameter 't' from ④, we get a relation involving c_1 & c_2.

Finally we replace c_1 & c_2 with the help of ② and obtain the required integral surface.

Method(2): we want to find the integral surface passing through the given curve which is determined by the following eqns $\phi(x,y,z)=0$ & $\psi(x,y,z)=0$ ——⑤

Now we eliminate x, y, z from the four eqns ② & ⑤ and obtain a relation between c_1 & c_2.

Finally, replace c_1 by $u(x,y,z)$ & c_2 by $v(x,y,z)$ in that relation and obtain the required integral surface.

Problems (Based on second method)

→ Find the integral of the PDE $(x-y)p+(y-x-z)q=z$ through the circle $z=1,\ x^2+y^2=1$

Sol^n: Given that $(x-y)p+(y-x-z)q=z$ —— ①

Lagrange's A.Es are

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z} \text{ —— ②}$$

Using the multipliers 1, 1, 1.

each fraction of ② $=\frac{dx+dy+dz}{0}$

$\Rightarrow dx+dy+dz=0$

$\Rightarrow \boxed{x+y+z=c_1}$ —— ③

Taking last two eqns of ②

$$\frac{dy}{y+y-c_1}=\frac{dz}{z}$$ [from ③ $x+y+z=c_1 \Rightarrow y-c_1=-x-z$]

$$\frac{dy}{2y-c_1}=\frac{dz}{z}$$

$$\frac{1}{2}\log(2y-c_1)=\log z+\log c$$

$$\log(2y-c_1)=\log z^2c^2$$

$$2y-c_1=z^2c_2 \quad \text{where } c^2=c_2$$

$$\Rightarrow \frac{2y-(x+y+z)}{z^2}=c_2 \quad (\text{from ③})$$

$$\Rightarrow \frac{y-x-z}{z^2}=c_2 \text{ —— ④}$$

The curve is given by $z=1,\ x^2+y^2=1$.

Taking $z=1$ in ③ & ④, we get

$x+y=c_1-1$ —— ⑤ & $y-x=c_2+1$ —— ⑥

But $2(x^2+y^2) = (x+y)^2 + (y-x)^2$ — (7)

Now using (5) & (6) in (7), we get

$2(1) = (c_1-1)^2 + (c_2+1)^2$

$\Rightarrow 2 = c_1^2 + c_2^2 - 2c_1 + 2c_2 + 2$

$\Rightarrow c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0$ — 8

putting the values c_1 & c_2 in (8), we get

$$(x+y+z)^2 + \frac{(y-x-z)^2}{z^4} - 2(x+y+z) + 2\frac{(y-x-z)}{z^2} = 0$$

$$\Rightarrow z^4(x+y+z)^2 + (y-x-z)^2 - 2z^4(x+y+z) + 2z^2(y-x-z) = 0$$

→ Find the eqn of the integral surface of the diff. eqn $(x^2-yz)p + (y^2-zx)q = z^2-xy$ — (1) which passes through the line $x=1,\ y=0$

Soln:

$\frac{x-y}{y-z} = c_1$ — (2)

$xy+yz+zx = c_2$ — (3)

The given curve is $x=1,\ y=0$ — (4)

using (4) in (2) & (3) we get

$\boxed{-\frac{1}{z} = c_1}$ $\boxed{z = c_2}$ — (5)

From (5) $\left(-\frac{1}{z}\right)(z) = c_1 c_2$

$\Rightarrow \boxed{c_1 c_2 = -1}$ — (6)

Using (2) & (3) in (6), we get

$$\left(\frac{x-y}{y-z}\right)(xy+yz+zx) = -1$$

Q7 → Find the eqn of surface satisfying $4yzp + q + 2y = 0$ and passing through $y^2+z^2=1$; $x+z=2$

Q8 (Method) → Find the integral surface of the linear PDE $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contains the straight line $x+y=0,\ z=1$ — (1)

Sol^n: $xyz = c_1$ —② ; $x^2+y^2-2z = c_2$ —③

Method-I

The given curve $x+y=0$ & $z=1$ —④

Taking t as a parameter

put $x=t$ in ④, we get

$y=-t$ and $z=1$

$\therefore x=t$; $y=-t$; $z=1$ —⑤

Using ⑤ in ② & ③, we get

$t(-t)(1)=c_1$ & $t^2+t^2-2=c_2$

$\Rightarrow -t^2=c_1$ & $2t^2-2=c_2$

$\Rightarrow t^2=-c_1$

$\Rightarrow 2(-c_1)-2=c_2$

$\Rightarrow 2c_1+c_2+2=0$ —⑥

Using ② and ③ in ⑥, we get

$2(xyz)+x^2+y^2-2z+2=0$

$\Rightarrow x^2+y^2+2xyz-2z+2=0$

(or)

2nd Method

Now eliminating x, y, z from ②, ③ & ④

we get $xy=c_1$ & $x^2+y^2-2=c_2$

$\Rightarrow (x+y)^2-2xy-2=c_2$

$\Rightarrow 0-2(c_1)-2=c_2$

$\Rightarrow 2c_1+c_2+2=0$

$\therefore$ From ② & ③, we get

$$x^2+y^2-2z+2xyz+2=0$$

2008 12M → Find the general solution of P.D.E

$(2xy-1)p+(z-2x^2)q = 2(x-yz)$

and also find the particular solution which passes through the lines $x=1$, $y=0$.

2005 → Find the particular integral of $x(y-z)p + y(z-x)q = z(x-y)$ which represents a surface passing through $x=y=z$.

Sol Given eqn is

$$x(y-z)p + y(z-x)q = z(x-y) \quad \text{---(1)}$$

Lagrange's A.E's are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \text{---(2)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get each fraction of (2) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow \log(xyz) = \log c_1$

$\Rightarrow \boxed{xyz = c_1}$ ---(2)

Again taking the multipliers are 1, 1, 1, we get each fraction of (2) $= \dfrac{dx+dy+dz}{0}$

$\therefore dx+dy+dz = 0$

$\Rightarrow \boxed{x+y+z = c_2}$ ---(3)

but curve is $x=y=z$ ---(4)

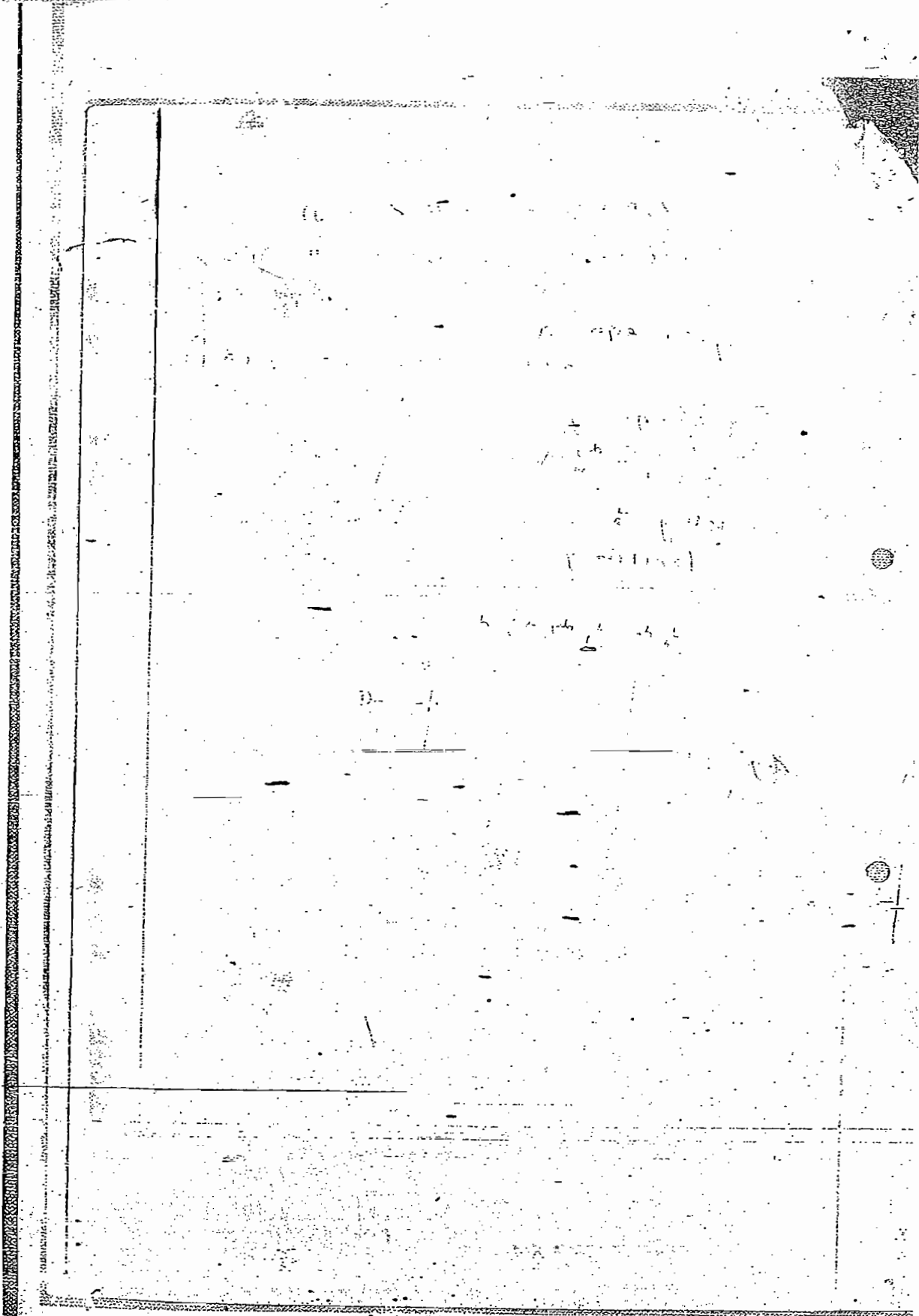
Now we eliminate x, y, z from (2), (3) & (4) we get

$x^3 = c_1$ & $3x = c_2$

$\Rightarrow \left(\frac{c_2}{3}\right)^3 = c_1$

$\Rightarrow c_2^3 - 27c_1 = 0$

$\Rightarrow (x+y+z)^3 - 27(xyz) = 0$

which is reqd surface of (1)

Charpit's Method:

We now give a general method due to Charpit for finding the complete integral of a non-linear differential eqn of the first order.

Let the given eqn be $f(x,y,z,p,q)=0$ —— ①

Since z depends on x & y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\Rightarrow dz = p\,dx + q\,dy \quad \text{—— ②}$$

The fundamental idea in charpit's method is the Introduction of another PDE of the first order

$$g(x,y,z,p,q,a)=0 \quad \text{—— ③}$$

which contains arbitrary constant 'a' and

(i) we can solve the eqns ① & ③ for

$$p = p(x,y,z,a) \;\&\; q = q(x,y,z,a)$$

(ii) Substituting these values of p & q in ②, the eqn ② becomes

$$dz = p(x,y,z,a)\,dx + q(x,y,z,a)\,dy \quad \text{—— ④}$$

This gives the solution, provided ④ is integral. If such a relation ③ has been found, the solution of the eqn ④

$$\phi(x,y,z,a,b)=0 \quad \text{—— ⑤}$$

containing two arbitrary constants a & b will be a solution of eqn ①.

Also it is a complete integral of the eqn ①.

How to determine g

Differentiating ① & ③ w.r.t x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = 0$$

and $$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x} = 0$$

Set-II

Non-linear Eqns :-

The integrals or solutions of the non-linear partial differentiable eqns of the first order:

W.K.T the relation of the type $f(x,y,z,a,b)=0$ — ①
gives rise to a PDE of the first order of the form
$F(x,y,z,p,q) = 0$. — ②
On the elimination of arbitrary constants a & b. Here x, y are independent variables and z is dependent variable.

→ If ① has been derived from ② then ① is a solution of ②.
Any such relation ① which contains as many arbitrary constants as there are independent variables, is called the complete integral or complete solution of ②.

→ Any particular integral of ② is obtained by giving particular values to a & b in ①.

Singular Integral (S.I):

The singular integral is obtained by eliminating a & b — b/w the three eqns $f(x,y,z,a,b)=0$, $\frac{\partial f}{\partial a}=0$ and $\frac{\partial f}{\partial b}=0$.

General Integral (G.I):

If in the eqn ①, one of the constants is a function of the other say $b=\phi(a)$ then ① becomes
$f(x,y,z,a,\phi(a))=0$ — ③
It is a one-parameter subfamily of the family (1).
The eqn of the envelope of the family of surfaces represented by ③ is also a solution of the eqn ②.
It is called the general integral of ② corresponding to the complete integral (1).

The eqn of the envelope of the surfaces represented by ③ is obtained by eliminating 'a' between the eqns
$f(x,y,z,a,\phi(a))=0$ and $\frac{\partial f}{\partial a}=0$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = 0$$

and $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} p + \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x} = 0$ ——— (6)

Again diff (1) & (3) w.r.t y, we get

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y} = 0$$

and $\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} q + \frac{\partial g}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0$ ——— (7)

Now eliminating $\frac{\partial p}{\partial x}$ from the eqns in (6) &

$\frac{\partial q}{\partial y}$ from the eqns in (7), we get

$$\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right) + p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial z}\right) + \frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right) = 0 \quad \text{—— (8)}$$

and $\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial y}\right) + q\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial z}\right) + \frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right) = 0$ —— (9)

Since $\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$$= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial p}{\partial y}$$

$$\therefore \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$$

(8) + (9) $\Rightarrow$

$$\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right) + p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial z}\right) +$$

$$\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial y}\right) + q\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial z}\right) = 0 \quad \left(\because \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}\right)$$

$$\Rightarrow \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial z}$$

$$+ \left(-\frac{\partial f}{\partial p}\right)\frac{\partial g}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial y} = 0 \quad \text{—— (10)}$$

Clearly it is a linear PDE of the first order with x, y, z, p, q as independent variables and g as a dependent variable.

The Lagrange's auxiliary eqns are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\partial f/\partial q} = \frac{d\phi}{0} \quad (11)$$

These eqns are known as Charpit's auxiliary eqns. Any of integrals of (11) satisfies (10). If such an integral contains p or q (or both) it can be taken the required second PDE (3).

Note: It should be noted that not all of Charpit's eqns (11) need be used, but that p or q must occur in the solution obtained.

Working Rule of Charpit's Method:

Step 1: Transfer terms of the given eqn to LHS and denote the entire expression by f.

Step 2: Write down the Charpit's auxiliary eqns (11).

Step 3: Using the value of f in step 1, write down the values of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc. occuring in step (2) and put these values in Charpit's auxiliary eqns (11).

Step 4: After simplifying step (3) select two proper fractions so that the resulting integral may come out to be the simplest relation involving atleast one of p and q.

Step 5: The simplest relation of step (4) is solved along with the given eqn to determine p & q.

Step 6: Put these values of p & q in
$dz = p\,dx + q\,dy$
which on integration gives the complete integral of the given eqn.

Problems:

→ Find the complete integral of $px+qy=pq$.

Sol: Given that $px+qy=pq$

$$\Rightarrow px+qy-pq=0$$

Let $f(x,y,z,p,q) = px+qy-pq=0$ ——— ①

Charpit's auxiliary eqns are

$$\frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-x+q} = \frac{dy}{-y+p} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+p(0)} = \frac{dq}{q}$$ ——— ②

Taking last two fractions of ②, we get

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a$$

$$\Rightarrow \boxed{p/q = a} \Rightarrow \boxed{p = qa}$$ ——— ③

① ≡ $qax+qy-(qa)q=0$

$$\Rightarrow q[ax+y-qa]=0$$

$$\Rightarrow ax+y-qa=0 \quad (\because q\neq 0)$$

$$\Rightarrow ax+y=qa$$

$$\Rightarrow \boxed{q = \frac{ax+y}{a}}$$ ——— ④

③ ≡ $p = \left(\frac{ax+y}{a}\right)a$

$$\Rightarrow \boxed{p = ax+y}$$ ——— ⑤

putting these values p and q in $dz = pdx+qdy$,

we get $dz = (ax+y)dx + \left(\frac{ax+y}{a}\right)dy$

$$\Rightarrow a\,dz = (ax+y)(a\,dx+dy)$$

$$\Rightarrow a\,dz = (ax+y)\,d(ax+y)$$

$$\Rightarrow az = \frac{(ax+y)^2}{2}+b$$

which is the complete integral of ①.

2000 → Solve by Charpit's method eqn

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0$$

Sol^n Let $f(x,y,z,p,q)=$

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0 \quad \text{---(1)}$$

The Charpit's auxiliary eqns are

$$\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dz}{-pf_p-qf_q}=\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z} \quad \text{---(2)}$$

From (1), $f_x=p^2(2x-1)+2pqy-2pz$

$f_y=2pqx+q^2(2y-1)-2qz$

$f_z=-2px-2qy+2z$

$f_p=2px(x-1)+2qxy-2xz$

$f_q=2pxy+2qy(y-1)-2yz$

and $f_x+pf_z=-p^2$; $f_y+qf_z=-q^2$

$$\therefore (2)\equiv \frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)}$$

$$=\frac{dz}{-p[2px(x-1)+2qxy-2xz]-q[2pxy+2qy(y-1)-2yz]}$$

$$=\frac{dp}{-p^2}=\frac{dq}{-q^2} \quad \text{---(3)}$$

each fraction of (3) $=\dfrac{\frac{1}{p}dp}{-p}=\dfrac{\frac{1}{q}dq}{-q}=\dfrac{\frac{1}{p}dp-\frac{1}{q}dq}{-p+q}$ ---(4)

Also each fraction of (3) $=\dfrac{\frac{1}{x}dx-\frac{1}{y}dy}{-2px+2p-2qy+2z+2px+2qy-2q-2z}$ ---(5)

$\therefore$ (4) & (5)

$$\Rightarrow \frac{\frac{1}{p}dp-\frac{1}{q}dq}{-(p-q)}=\frac{\frac{1}{x}dx-\frac{1}{y}dy}{2(p-q)}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{x}dx-\frac{1}{y}dy\right)=\frac{1}{q}dq-\frac{1}{p}dp$$

Integrating, we get

$$\frac{1}{2}(\log x-\log y)=\log q-\log p+\log a.$$

$$\Rightarrow \left(\frac{x}{y}\right)^{1/2} = \frac{q}{p}\cdot a$$

$\therefore$ $\boxed{p = \frac{a y^{1/2} q}{x^{1/2}}}$; a is arbitrary constant. — (6)

$\therefore$ (1) $\equiv (px+qy-z)^2 = p^2x + q^2y$

$$\Rightarrow px+qy-z = \pm\sqrt{p^2x+q^2y} \quad \text{—— (7)}$$

Taking +ve sign in (7)

$$px+qy-z = \sqrt{p^2x+q^2y} \quad \text{—— (8)}$$

$$\Rightarrow \left(\frac{a y^{1/2} q}{x^{1/2}}\right)x + qy - z = \sqrt{\left(\frac{a^2 y q^2}{x}\right)(x) + q^2 y} \quad (\text{by } (6))$$

$$\Rightarrow aq(xy)^{1/2} + qy - z = \sqrt{ya^2q^2+q^2y} = qy^{1/2}(1+a^2)^{1/2}$$

$$\Rightarrow q\left[y + a(xy)^{1/2} - (1+a^2)^{1/2}y^{1/2}\right] = z$$

$$\Rightarrow \boxed{q = \frac{z}{y + a(xy)^{1/2} - (1+a^2)^{1/2}y^{1/2}}} \quad \text{—— (9)}$$

putting these values in $dz = pdx + qdy$

$$dz = \frac{az\,dx}{x^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]} + \frac{z\,dy}{y^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]}$$

$$\Rightarrow \frac{dz}{z} = \frac{ay^{1/2}dx + x^{1/2}dy}{(xy)^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]}$$

Integrating

$$\Rightarrow \log z = 2\log\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right] + \log b$$

$$\Rightarrow z = b\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]^2; \quad b \text{ is an arbitrary constant.}$$

2002 → Solve $z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$

2002 → Find two complete integrals of the PDE $x^2p^2+y^2q^2-4=0$

Soln: Given that $x^2p^2+q^2y^2-4=0$

Let $f(x,y,z,p,q) = x^2p^2 + y^2q^2 - 4 = 0$ —— ①

∴ Charpit's A.Es are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\Rightarrow \frac{dx}{-2x^2p} = \frac{dy}{-2y^2q} = \frac{dz}{-p(2x^2p) - q(2y^2q)} = \frac{dp}{2xp^2} = \frac{dq}{2yq^2} \quad \text{—— ②}$$

To find first complete integral:

Taking the first & fourth fraction of ②,

we get $\frac{dx}{-2x^2p} = \frac{dp}{2xp^2} \Rightarrow \frac{dx}{-x} = \frac{dp}{p}$

$\Rightarrow \log x = -\log p + \log c$

$\Rightarrow \log(px) = \log c$

$\Rightarrow px = c \Rightarrow \boxed{p = \frac{c}{x}}$

$① \equiv x^2 \frac{c^2}{x^2} + y^2q^2 = 4$

$\Rightarrow c^2 + y^2q^2 = 4 \Rightarrow q^2 = \frac{4-c^2}{y^2}$

$\Rightarrow \boxed{q = \frac{\sqrt{4-c^2}}{y}}$

Substituting the values of p & q in

$dz = pdx + qdy$, we get

$\Rightarrow dz = \frac{c}{x}dx + \frac{\sqrt{4-c^2}}{y}dy$

Integrating

$\boxed{z = c\log x + \sqrt{4-c^2}\log y + \log b}$

Taking 2nd & last fractions of ②, we get

$\frac{dy}{-2y^2q} = \frac{dq}{2yq^2} \Rightarrow \frac{dy}{-y} = \frac{dq}{q}$

$\Rightarrow \log y = -\log q + \log d$

$\Rightarrow yq = d$

$\Rightarrow \boxed{q = \frac{d}{y}}$

$∴ ① \equiv x^2p^2 + d^2 = 4$

$\Rightarrow p^2 = \frac{4-d^2}{x^2} \Rightarrow \boxed{p = \frac{\sqrt{4-d^2}}{x}}$

Substituting the values of p & q in

$dz = p\,dx + q\,dy$.

$$\Rightarrow dz = \frac{\sqrt{4-d^2}}{x}dx + \frac{d}{y}dy$$

$$\Rightarrow \boxed{z = \sqrt{4-d^2}\log x + d\log y}$$

→ Find three complete integrals of $pq = px + qy$.

2004 → Find a complete, singular, and general integrals of $(p^2+q^2)y = qz$

Sol^n: Given that $(p^2+q^2)y - qz = 0$

Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ ——①

A Eqns are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\Rightarrow \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dp}{-pq} = \frac{dq}{p^2}$$ ——②

Taking last two fractions of ②

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$$\Rightarrow p\,dp + q\,dq = 0$$

$$\Rightarrow \boxed{p^2+q^2 = a^2}$$ ——③

①≡ $a^2y = qz$

$$\Rightarrow \boxed{q = \frac{a^2y}{z}}$$

③≡ $p^2 + \frac{a^4y^2}{z^2} = a^2$

$$\Rightarrow p^2 = a^2 - \left(\frac{a^2y}{z}\right)^2$$

$$\Rightarrow p = \sqrt{a^2 - \left(\frac{a^2y}{z}\right)^2}$$

$$\Rightarrow \boxed{p = \frac{a}{z}\sqrt{z^2 - a^2y^2}}$$

Putting these p and q in $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

$$\Rightarrow (z^2-a^2y^2)^{1/2} = ax+b$$

$$\Rightarrow z^2-a^2y^2 = (ax+b)^2 \quad \text{---} (4)$$

which is the required complete integral.

Singular integral:

Diff (4) w.r.t a & b, we get

$$-2ay^2 = 2(ax+b)x$$

$$\Rightarrow \boxed{ax^2+bx+ay^2=0} \quad \text{---} (5)$$

and $2(ax+b)=0$

$$\Rightarrow \boxed{ax+b=0} \quad \text{---} (6)$$

Now eliminating a & b from (4), (5) & (6), we get

(5) $\equiv x(0)+ay^2=0$ (by (6))

$$\Rightarrow \boxed{a=0} \quad (\because y\neq 0)$$

(6) $\equiv \boxed{b=0}$

$\therefore$ (4) $\equiv z=0$ which clearly satisfies (1)

$\therefore$ It is the required singular solution of (1).

General Integral:

Let $b=\phi(a)$ in (4), then $z^2-a^2y^2=[ax+\phi(a)]^2$ (7)

Diff (7) partially w.r.t a, we get

$$-2ay^2 = 2(ax+\phi(a))(x+\phi'(a)) \quad \text{---} (8)$$

$\therefore$ G.I is obtained by eliminating 'a' from (7) & (8).

1997 → Find a complete integral of $z^2(p^2z^2+q^2)=1$

1996 → Find a complete integral of $z=px+qy+p^2+q^2$

1994 → Find a complete integral of $16p^2z^2+9q^2+4z^2-4=0$

1998 → " " " " $2x(z^2q^2+1)=pz$

1999 → " " " " $p^2+q^2-2px-2qy+1=0$

Special Types of equations:

we shall consider some special types of first-order partial differential eqns whose solutions may be obtained easily by Charpit's Method.

Type 1: Equations involving only p & q :-

For eqns of the type $f(p,q)=0$ —— ①

Charpit's auxiliary eqns are

$$\frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{0} = \frac{dq}{0}$$

Taking third & fourth fractions, we get

$dp=0 \Rightarrow \boxed{p=a}$ (constant) —— ②

① $\Rightarrow f(a,q)=0$

$\Rightarrow q = \text{constant}$ —— ③

$= \phi(a)$ (say)

putting these values in $dz = p\,dx + q\,dy$

$\Rightarrow dz = a\,dx + \phi(a)\,dy$

integrating

$z = ax + \phi(a)y + b$ —— ④

where b is constant

which is a complete integral of ①

It contains two arbitrary constants a & b.

General Integral:

putting $b=\psi(a)$ in ④, where ψ is arbitrary function

we get,

$z = ax + \phi(a)y + \psi(a)$ —— ⑤

Diff ⑤ partially w.r.t 'a', we get-

$0 = x + \phi'(a) \cdot y + \psi'(a)$ —— ⑥

eliminating a b/w ⑤ & ⑥.

Singular Integral: The singular integral, if it exists, is obtained by eliminating a & b between the complete integral ④ and the eqns formed by differentiating ④ partially w.r.t a & b.

i.e, b/w the eqns

$z = ax + \phi(a) y + b$,

$0 = x + \phi'(a) y$ and $0 = 1$

Since $1 = 0$ is inconsistent (meaningless)

∴ In this case there is no singular solution.

→ Solve $pq = k$, where k is constant.

Sol: The given eqn is $pq = k$ —— ①

where k is constant.

Clearly the eqn ① is of the form $f(p, q) = 0$

∴ Its complete integral is

$z = ax + \phi(a) y + b$ —— ②

Taking $a = p$; $\phi(a) = q$

∴ ① ≡ $a\,\phi(a) = k$

$\Rightarrow \phi(a) = \frac{k}{a}$

∴ ② ≡ $z = ax + \frac{k}{a} y + b$ —— ③

where a & b are arbitrary constants.

∴ It is the required complete integral.

To find singular integral:

Diff ③ partially w.r.t a & b, we get

$0 = x - \frac{k}{a^2} y$

$0 = 1$ which is meaningless

$\therefore$ The given eqn ③ has no singular integral

General integral:

putting $b = \phi(a)$ in ③, we get

$$z = ax + \frac{k}{a}y + \phi(a) \quad \text{--- ④}$$

diff ④ partially w.r.t a, we get

$$0 = x - \frac{k}{a^2}y + \phi'(a) \quad \text{--- ⑤}$$

$\therefore$ The required G.I is obtained by eliminating a between ④ & ⑤.

→ Solve $q = 3p^2$

→ Solve $q = e^{-p/\alpha}$ where α is a constant.

→ Find the complete integral of $\sqrt{p} + \sqrt{q} = 1$

Equations reducible to type 1:

→ Find the complete integral of

$$z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0 \quad \text{--- ①}$$

Sol^n ① ≡ $z^2y\left(\frac{\partial z}{\partial x}\right)^2 + 6zxy\left(\frac{\partial z}{\partial x}\right) + 2zx^2\left(\frac{\partial z}{\partial y}\right) + 4x^2y = 0$

Dividing by x^2y, we get

$$\Rightarrow \frac{z^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{6z}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{2z}{y}\left(\frac{\partial z}{\partial y}\right) + 4 = 0$$

$$\Rightarrow \left(\frac{z}{x}\frac{\partial z}{\partial x}\right)^2 + 6\left(\frac{z}{x}\frac{\partial z}{\partial x}\right) + 2\left(\frac{z}{y}\frac{\partial z}{\partial y}\right) + 4 = 0 \quad \text{②}$$

putting $x\,dx = dX$; $y\,dy = dY$; $z\,dz = dZ$

$$\Rightarrow \frac{x^2}{2} = X; \quad \frac{y^2}{2} = Y; \quad \frac{z^2}{2} = Z$$

$$② \equiv \left(\frac{\partial Z}{\partial X}\right)^2 + 6\left(\frac{\partial Z}{\partial X}\right) + 2\left(\frac{\partial Z}{\partial Y}\right) + 4 = 0$$

$$\Rightarrow P^2 + 6P + 2Q + 4 = 0$$

where $P = \frac{\partial Z}{\partial X}$, $Q = \frac{\partial Z}{\partial Y}$

Clearly ③ is in the form of $f(P,Q)=0$

∴ Its complete integral is of the form

$$Z = ax + \phi(a)y + b \quad \text{—④}$$

where $a = P$ & $\phi(a) = Q$.

$$③ \equiv a^2 + 6a + 2\phi(a) + 4 = 0$$

$$\Rightarrow \phi(a) = -\frac{(a^2+6a+4)}{2}$$

$$④ \equiv Z = ax - \left(\frac{a^2+6a+4}{2}\right)Y + b$$

where a & b are arbitrary constants.

$$\therefore \frac{z^2}{2} = a\left(\frac{x^2}{2}\right) - \left(\frac{a^2+6a+4}{2}\right)\left(\frac{y^2}{2}\right) + b.$$

which is the required complete integral of ①

→ Solve $x^2p^2 + y^2q^2 = z^2$ —①

$$\Rightarrow \frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\Rightarrow \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\Rightarrow \left(\frac{\frac{1}{z}\partial z}{\frac{1}{x}\partial x}\right)^2 + \left(\frac{\frac{1}{z}\partial z}{\frac{1}{y}\partial y}\right)^2 = 1 \quad \text{—②}$$

putting $\frac{1}{x}dx = dX$; $\frac{1}{y}dy = dY$; $\frac{1}{z}dz = dZ$

$\Rightarrow \boxed{\log x = X}$; $\boxed{\log y = Y}$; $\boxed{\log z = Z}$

$$\therefore ② \equiv \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

$$\Rightarrow P^2 + Q^2 = 1 \quad \text{—③}$$

∴ It is of the form $f(P,Q)=0$

Its complete integral is of the form

$$Z = aX + \phi(a)Y + b \quad \text{—④}$$

Taking $a = P$ & $\phi(a) = Q$

$$\therefore (3) \equiv a^2 + [\phi(a)]^2 = 1$$

$$\Rightarrow [\phi(a)]^2 = 1 - a^2$$

$$\Rightarrow \phi(a) = \sqrt{1-a^2}$$

$$\therefore (4) \equiv z = ax + (\sqrt{1-a^2})y + b$$

where a & b are arbitrary constants.

$$\Rightarrow \log z = a \log x + (\sqrt{1-a^2}) \log y + b$$

which is the required complete integral.

If we take

$$a = \cos\alpha, \quad b = \log c$$

then complete integral is

$$\log z = \cos\alpha \log x + \sin\alpha \log y + \log c.$$

$$\Rightarrow \boxed{z = x^{\cos\alpha} y^{\sin\alpha} c} \quad \text{—(5)}$$

where α & c are arbitrary constants

Singular Integrals

Diff (5) partially w.r.t α & c, we get

$$0 = c\left[x^{\cos\alpha}\left(y^{\sin\alpha} \log y\right)\cos\alpha + y^{\sin\alpha} x^{\cos\alpha} \log x \cdot (-\sin\alpha)\right]$$

$$\Rightarrow x^{\cos\alpha} y^{\sin\alpha}(\cos\alpha \log y - \sin\alpha \log x) = 0 \quad \text{—(6)}$$

and $0 = x^{\cos\alpha} y^{\sin\alpha}$ —(7)

Eliminating α, c from (1), (5), (6), (7); we get

$$z = 0$$

which is the required singular solution.

G.I: putting $c = \phi(\alpha)$

$$(5) \equiv z = x^{\cos\alpha} y^{\sin\alpha} \phi(\alpha) \quad \text{—(8)}$$

Diff (8) partially w.r.t α.

$$0 = x^{\cos\alpha}\,\phi(\alpha)\left(y^{\sin\alpha}\log y \cdot \cos\alpha\right) + x^{\cos\alpha}\,y^{\sin\alpha}\,\phi'(\alpha) \quad \text{--- (9)}$$

Eliminating α from (8) & (9), we get the required G.I of (1).

→ Find a complete integral of

(i) $pq = x^m y^n z^{2l}$ (ii) $pq = x^m y^n z^l$

Sol: (i) Given that

$$pq = x^m y^n z^{2l} \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = x^m y^n z^{2l}$$

$$\Rightarrow \left(\frac{z^{-2l}}{x^m y^n}\right)\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

$$\Rightarrow z^{-l}dz = dZ \;;\; x^m dx = dX \;;\; y^n dy = dY$$

$$\Rightarrow \boxed{Z = \frac{z^{-l+1}}{-l+1}} \;;\; \boxed{X = \frac{x^{m+1}}{m+1}} \;;\; \boxed{Y = \frac{y^{n+1}}{n+1}}$$

Type 2: Eqns not involving the independent variables:

If the partial diff. eqn is of the type $f(z, p, q) = 0$ --- (1)

Charpit's auxiliary eqns reduce to

$$\frac{dx}{\partial f/\partial p} = \frac{dy}{\partial f/\partial q} = \frac{dz}{p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}} = \frac{dp}{-p\frac{\partial f}{\partial z}} = \frac{dq}{-q\frac{\partial f}{\partial z}} \quad \text{--- (2)}$$

Taking last two fractions, we get

$$\frac{dq}{q} = \frac{dp}{p} \Rightarrow \frac{q}{p} = a$$

$$\Rightarrow \boxed{q = pa} \quad \text{--- (3)}$$

where a is arbitrary constant.

∴ from $dz = p\,dx + q\,dy$, we get

$$dz = p(dx + a\,dy)$$

$\Rightarrow dz = p\,d(x+ay)$

$\Rightarrow dz = p\,dX$ where $X = x+ay$

$\Rightarrow \boxed{\frac{dz}{dX} = p}$

$\therefore$ ③ $\equiv \boxed{q = a\frac{dz}{dX}}$

$\therefore$ ① $\equiv f\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0$

which is an ordinary diff. eqn of the first order and solving it, a complete integral can be obtained.

Working rule:-

Step (1): write down the given eqn $f(p,q,z)=0$

Step (2): put $p = \frac{dz}{dX}$ & $q = a\frac{dz}{dX}$ where $X = x+ay$

Step (3): solving the resulting ODE in the variables z & X then substitute $X = x+ay$. This gives the complete integral.

Note: Some times using transformations eqns to the form of type (II).

$\rightarrow$ Find a complete integral of $9(p^2z+q^2)=4$

Sol: Given that $9(p^2z+q^2)=4$ —— ①

Clearly it is of the form $f(p,q,z)=0$

where $p = \frac{dz}{dX}$, $q = a\frac{dz}{dX}$

① $\equiv 9\left[\left(\frac{dz}{dX}\right)^2 z + a^2\left(\frac{dz}{dX}\right)^2\right] = 4$

where $X = x+ay$ —— ②

$\Rightarrow 9(z+a^2)\left(\frac{dz}{dX}\right)^2 = 4$

$$\Rightarrow \frac{dz}{dx} = \frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow \frac{3}{2}\sqrt{z+a^2}\,dz = dx$$

$$\Rightarrow \frac{3}{2}\frac{(z+a^2)^{3/2}}{3/2} = x+b$$ where b is arbitrary constant

$$\Rightarrow (z+a^2)^{3/2} = (x+ay)+b \quad (\text{by } ②)$$

$$\Rightarrow (z+a^2)^3 = (x+ay+b)^2$$

where a & b are arbitrary constants.

which is the required complete integral.

→ Find the complete integral of the eqn $p^2z^2+q^2=1$

1991 → Find a complete integral of $p^3+q^3-3pqz=0$

1993 → Find a complete integral of $z^2(p^2+q^2+1)=k^2$ where k is constant.

→ Find a complete integral of $q^2y^2 = z(z-px)$. ——①

Hint: $\left(y\cdot\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right)$

$$\Rightarrow \left(\frac{\partial z}{\frac{1}{y}\partial y}\right)^2 = z\left(z - \frac{\partial z}{\frac{1}{x}\partial x}\right) \text{——②}$$

Taking $\frac{1}{x}dx = dX$ ‖ $\frac{1}{y}dy = dY$

$\log x = X$ ‖ $\log y = Y$.

$$\therefore ① \equiv \left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right)$$

$$\Rightarrow z(z-P) = Q^2$$

where $P = \frac{\partial z}{\partial X}$; $Q = \frac{\partial z}{\partial Y}$

and proceed.

Type(III)

Seperable equations

Eqns not involving z and it happens the terms containing p & x can be seperated from those containing q & y

i.e, they have the form

$$f_1(x, p) = f_2(y, q) \quad \text{—— (1)}$$

Corresponding Charpit's auxiliary eqns are

$$\frac{dx}{\frac{\partial f_1}{\partial p}} = \frac{dy}{-\frac{\partial f_2}{\partial q}} = \frac{dp}{-\frac{\partial f_1}{\partial x}} = \frac{dq}{-\frac{\partial f_2}{\partial y}} \quad \text{—— (2)}$$

$$\frac{\partial f_1}{\partial p} dp + \frac{\partial f_1}{\partial x} dx = 0$$

$$\Rightarrow df_1 = 0 \quad \left(\because df_1 = \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial p} dp\right)$$

$$\Rightarrow f_1 = \text{constant}$$

$$\Rightarrow f_1(x, p) = a \text{ (say)} \quad \text{—— (3)}$$

$$\therefore (1) \equiv f_2(y, q) = f_1(x, p)$$

$$\Rightarrow f_2(y, q) = a \quad \text{—— (4)}$$

solving (3) & (4) for p & q, we get —

$$p = F_1(x, a), \quad q = F_2(y, a)$$

putting these values of p & q in $dz = p\,dx + q\,dy$,

we get $dz = F_1(x, a)\,dx + F_2(y, a)\,dy$

Integrating we get

$$z = \int F_1(x, a)\,dx + \int F_2(y, a)\,dy + b$$

where b is an arbitrary constant.

: which is the required complete integral.

working rule:

Step1: write the given eqn in the form $f_1(x, p) = f_2(y, q)$.

Step2: putting both sides of the above eqn equal

to an arbitrary constant. we get the two eqns.

Step 3: Solving them for p & q.

Substitute the values of p & q in

$dz = p\,dx + q\,dy$.

Integrate, we get the complete integral of ①

Note: Some times using transformations eqns reduce to the form of type III

→ Find a complete integral of $p^2 + q^2 = x + y$

Sol^n: Given that $p^2 + q^2 = x + y$ — ①

$\Rightarrow p^2 - x = y - q^2$ — ②

$\Rightarrow p^2 - x = y - q^2 = a$ (say).

$\Rightarrow p^2 - x = a$ — ③

& $y - q^2 = a$ — ④

③ ≡ $p = \sqrt{x+a}$ & ④ ≡ $q = \sqrt{y-a}$

∴ putting these values of p & q in $dz = p\,dx + q\,dy$

$\Rightarrow dz = \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$

Integrating

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b$$

which is the required complete integral.

1989 → Find a complete integral of $z^2(p^2+q^2) = x^2+y^2$.

Sol^n: i.e. $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2+y^2$ — ①

$\Rightarrow \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2$ — ②

Taking $z\,dz = dZ$

③ $\equiv \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = x^2+y^2$ –

$\Rightarrow P^2+Q^2 = x^2+y^2$ where $P = \frac{\partial z}{\partial x}$ $Q = \frac{\partial z}{\partial y}$

and proceed.

89 → find a complete integral of $z(p^2-q^2) = x-y$

2001 → find the complete integral of the PDE
$2p^2q^2+3x^2y^2 = 8x^2q^2(x^2+y^2)$.

Sol: Given that

$2p^2q^2+3x^2y^2 = 8x^2q^2(x^2+y^2)$ — ①

$\Rightarrow 2q^2(p^2-4x^4) = x^2y^2(8q^2-3)$

$\Rightarrow \frac{p^2-4x^4}{x^2} = \frac{y^2(8q^2-3)}{2q^2} = 4a^2$ (say) (constant)

$\Rightarrow \frac{p^2-4x^4}{x^2} = 4a^2$; $\frac{y^2(8q^2-3)}{2q^2} = 4a^2$

$\Rightarrow p^2 = 4x^2(a^2+x^2)$; $8q^2(y^2-a^2) = 3y^2$

$\Rightarrow p = 2x(a^2+x^2)^{1/2}$; $q^2 = \frac{3y^2}{8(y^2-a^2)} = \left(\frac{1}{4}\right)\left(\frac{3}{2}\right)\frac{y^2}{y^2-a^2}$

$q = \left(\frac{3}{2}\right)^{1/2}\left(\frac{y}{2}\right)\frac{1}{(y^2-a^2)^{1/2}}$

substituting the values of p & q in
$dz = p\,dx + q\,dy$

$\Rightarrow dz = 2x(a^2+x^2)^{1/2}dx + \left(\frac{3}{2}\right)^{1/2}\left(\frac{y}{2}\right)(y^2-a^2)^{-1/2}$

Integrating

$\Rightarrow z = \frac{4}{3}(a^2+x^2)^{3/2} + \left(\frac{3}{2}\right)^{1/2}(y^2-a^2)^{1/2} + b.$

Type(IV) Clairaut eqn :-

A first order PDE is said to be Clairaut form if it is in the form $z = px + qy + f(p,q)$ —— (1)

The corresponding Charpit's auxiliary eqns are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$\Rightarrow p = a$ & $q = b$

$\therefore$ (1) $\equiv z = ax + by + f(a,b)$ —— (2)

which is the required complete integral.

→ To find the general integral

put $b = \phi(a)$ in (2), where ϕ is an arbitrary function.

then $z = ax + y\phi(a) + f\{a, \phi(a)\}$ —— (3)

Diff (3) partially w.r.t. a, we get

$0 = x + y\phi'(a) + f'(a)$ —— (4)

Eliminating a b/w (3) & (4), we get G.I of (1).

→ To find the S.I, eliminate a & b b/w the three eqns $z = ax + by + f(a,b)$

$x + \frac{\partial f}{\partial a} = 0$ and $y + \frac{\partial f}{\partial b} = 0$

Note: Some times, using the transformations eqns reduce to the form of type IV.

Problems: Find the singular integral of $z = px + qy + c\sqrt{1+p^2+q^2}$

Soln: Given $z = px + qy + c\sqrt{1+p^2+q^2}$

It is of the Clairaut's form.

$\therefore$ Its complete integral is $z = ax + by + c\sqrt{1+a^2+b^2}$ —— (1)

Singular integral:

Diff ① partially w.r.t a and b,

we get $0 = x + \frac{ac}{\sqrt{1+a^2+b^2}}$ —②, $0 = y + \frac{bc}{\sqrt{1+a^2+b^2}}$ —③

from ② & ③

$$x^2+y^2 = \frac{a^2c^2+b^2c^2}{1+a^2+b^2}$$

$$\Rightarrow c^2-x^2-y^2 = \frac{c^2}{1+a^2+b^2}$$

$$\Rightarrow 1+a^2+b^2 = \frac{c^2}{c^2-x^2-y^2} \quad \text{—④}$$

from② $a = -\frac{x\sqrt{1+a^2+b^2}}{c}$

$$= \frac{-x}{\sqrt{c^2-x^2-y^2}} \quad (\text{by } ④) \quad \text{—⑤}$$

and from③ $b = -\frac{y\sqrt{1+a^2+b^2}}{c}$

$$= \frac{-y}{\sqrt{c^2-x^2-y^2}} \quad \text{—⑥}$$

putting the values from ④, ⑤ & ⑥ in ①, the singular solution is

$$z = -\frac{x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}} = \sqrt{c^2-x^2-y^2}$$

$$\Rightarrow z^2 = c^2-x^2-y^2$$

$$\Rightarrow \boxed{x^2+y^2+z^2=c^2}$$

→ Find a complete and singular integral of $4xyz = pq + 2px^2y + 2qxy^2$.

Soln: Given that $4xyz = pq + 2p x^2 y + 2q xy^2$ —— (1)

$$\Rightarrow z = \frac{1}{4xy}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + \frac{1}{2xy}\left(\frac{\partial z}{\partial x}\right)x^2y + \frac{1}{2xy}\left(\frac{\partial z}{\partial y}\right)xy^2$$

$$\Rightarrow z = \left(\frac{1}{2x}\cdot\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) + \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)x^2 + \left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)y^2 \text{ —— (2)}$$

Taking $2x\,dx = dX$; $2y\,dy = dY$

$\Rightarrow \boxed{x^2 = X}$; $\Rightarrow \boxed{y^2 = Y}$

(2) ≡ $$z = \left(\frac{\partial z}{\partial X}\right)\left(\frac{\partial z}{\partial Y}\right) + \left(\frac{\partial z}{\partial X}\right)X + \left(\frac{\partial z}{\partial Y}\right)Y$$

$\Rightarrow z = PX + QY + PQ$ —— (3)

where $P = \frac{\partial z}{\partial X}$; $Q = \frac{\partial z}{\partial Y}$.

Clearly which is in Clairaut's form.

∴ The complete integral of (3) is

$z = aX + bY + ab$ (by putting $P = a$ & $Q = b$)

$\Rightarrow \boxed{z = ax^2 + by^2 + ab}$ —— (4)

which is the required complete integral of (1).

Singular integral:

Differentiating (4) w.r.t a & b, we get

$0 = x^2 + b \Rightarrow \boxed{b = -x^2}$ —— (5)

and $0 = y^2 + a \Rightarrow \boxed{a = -y^2}$ —— (6)

∴ (4) ≡ $z = -y^2x^2 - x^2y^2 + x^2y^2$

$\Rightarrow \boxed{z = -x^2y^2}$

which is the required singular integral of (1)

2008 15M → Find complete and singular integrals of $2xz - px^2 - 2qxy + pq = 0$ using Charpit's method.

Solutions satisfying given conditions :-

We shall consider the determination of surfaces which satisfy the P.DE $f(x, y, z, p, q) = 0$ —① and which satisfy some other condition as passing through given curve (or) circumscribing a given surface. we shall also consider how to derive one complete integral from another.

I → first of all, we shall discuss how to Determine the solution of ① which passes through a given curve 'c' which has parametric eqns

$x = x(t),\ y = y(t),\ z = z(t)$ —② where t is parameter.

⇒ If there is an integral surface of the eqn ① through the curve 'c', then it is;

(a) A particular case of the complete integral $f(x, y, z, a, b) = 0$ —③ obtained by giving particular values to a or b.

(or)

(b) A particular case of the general integral corresponding to ③.
i.e, the envelope of a one-parameter subfamily of ③.

(or)

(c) The envelope of the two-parameter system ③.

→ Now the points of intersection of the surface ③ and the curve 'c' are determined in terms of the parameter 't' by the eqn $f(x(t), y(t), z(t), a, b) = 0$ —④

and the condition that the curve 'c' should touch the

Surface ③ is that the eqn ④ must have two equal roots (i.e, $b^2-4ac=0$)

(or) the eqn ④ and

the eqn $\frac{\partial}{\partial t} f(x(t), y(t), z(t), a, b) = 0$ —— ⑤

should have a common root.

Now eliminating 't' from ④ & ⑤, we get the relation b/w a & b of the type $\psi(a,b)=0$ —— ⑥

The eqn ⑥ may be factorised into a set of alternative eqns.

$b=\phi_1(a),\ b=\phi_2(a), \ldots$ —— ⑦

∴ Each of which defines a subsystem of one-parameter.

The envelope of each of these one-parameter subsystems is a solution of the problem.

Problems

2004 Find a complete integral of the PDE $(p^2+q^2)x = pz$ and deduce the solution which passes through the curve $x=0, z^2=4y$.

Sol: Given that $(p^2+q^2)x = pz$.

Let $f(x,y,z,p,q) = (p^2+q^2)x - pz = 0$ —— ①

By Charpit's method its complete integral

$$z^2 = a^2x^2 + (ay+b)^2 \quad \text{—— ②}$$

and the given curve is $x=0,\ z^2=4y$ —— ③

Now taking 't' as parameter in ③.

we get $x=0,\ y=t^2,\ z=2t$ —— ④

The intersection of ② & ④ is

$$(2t)^2 = a^2(0) + (at^2+b)^2$$

$$\Rightarrow 4t^2 = a^2t^4 + b^2 + 2abt^2$$

$$\Rightarrow a^2(t^2)^2 + (2ab-4)t^2 + b^2 = 0 \quad \text{—— ⑤}$$

This has equal roots

if $(2ab - z_4)^2 - 4a^2b^2 = 0$

$\Rightarrow \; 4 - 4ab = 0$

$\Rightarrow \; \boxed{ab = 1}$ —— ⑥

$\therefore \; \boxed{b = \frac{1}{a}}$

$\therefore$ ③ $\equiv \; z^2 = a^2x^2 + \left(ay + \frac{1}{a}\right)^2$ —— ⑦

$\therefore$ which is the one parameter subsystem of ③

⑦ $\equiv \; x^2a^2 + y^2a^2 + \frac{1}{a^2} + 2y - z^2 = 0$

$\Rightarrow (x^2+y^2)\,a^4 + (2y - z^2)a^2 + 1 = 0$ —— ⑧

This has equal roots

if $(2y - z^2)^2 - 4(x^2+y^2) = 0$ —— ⑨

which required envelope of ⑧

⑨ $\equiv \; 2y - z^2 = 2\sqrt{x^2+y^2}$

$\Rightarrow \; z^2 = 2y - 2\sqrt{x^2+y^2}$

which ~~is the~~ required solution of the eqn ①.

→ Find a complete integral of the eqn $p^2x + qy = z$ and hence derive the eqn of an integral surface of which the line $y = 1$, $x + z = 0$ is a generator.

→ Show that integral surface of the eqn $z(1-q^2) = 2(px + qy)$ which pass through the line $x = 1$, $y = bz + k$ has the eqn $(y - kx)^2 = z^2\{(1+b^2)x - 1\}$.

[II] The problem of deriving one complete integral from another may be treated in a very similar way.

Suppose we know that $f(x, y, z, a, b) = 0$ —— ①

is complete integral of $F(x, y, z, p, q) = 0$ —— ②

and we want to show that another relation

$g(x, y, z, h, k) = 0$ —— ③

where h & k are arbitrary constants

is also complete integral of ②.

we choose on the surface ③ a curve c in whose eqns the constants h, k appear as independent parameters and then find the envelope of the one-parameter subfamily of ① touching the curve c.

Since this solution contains two arbitrary constants, it is a complete integral.

→ Show that the equation $xpq + yq^2 = 1$ has complete integrals (a) $(z+b)^2 = 4(ax+y)$

(b) $kx(z+h) = k^2y + x^2$

and deduce (b) from (a)

Soln: Given that $xpq + yq^2 = 1$ ——①

Charpit's auxiliary eqns are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\Rightarrow \frac{dx}{-xq} = \frac{dy}{-[xp+2yq]} = \frac{dz}{p(xq)+q(xp+2yq)} = \frac{dp}{pq} = \frac{dq}{q^2}$$ ——②

Now taking last two fractions from ②, we have

$$\boxed{p = qa}$$

① ≡ $xq^2a + yq^2 = 1$

$\Rightarrow q^2(xa+y) = 1$

$$\Rightarrow \boxed{q = \frac{1}{\sqrt{xa+y}}}$$

$$\boxed{\therefore p = \frac{a}{\sqrt{xa+y}}}$$

Substituting the values of p and q in $dz = pdx + qdy$

$$\Rightarrow dz = \frac{1}{\sqrt{xa+y}}[adx + dy]$$

$$\Rightarrow dz = \frac{d(ax+y)}{\sqrt{xa+y}}$$

$$\Rightarrow z = 2(xa+y)^{1/2} + b$$

$$\therefore \Rightarrow \boxed{(z+b)^2 = 4(ax+y)} \quad \text{--- (a)}$$

Taking first & last fractions of (2), we get

$$\frac{dx}{-x} = \frac{dq}{q}$$

$$\Rightarrow xq = k \Rightarrow \boxed{q = \frac{k}{x}}$$

$$\therefore (1) \equiv pk + y\frac{k^2}{x^2} = 1$$

$$\Rightarrow \boxed{p = \frac{1}{k}\left(\frac{x^2 - yk^2}{x^2}\right)}$$

$$\therefore dz = \left(\frac{x^2 - yk^2}{x^2 k}\right) dx + \frac{k}{x} dy$$

$$dz = \frac{1}{k} dx - \frac{yk}{x^2} dx + \frac{k}{x} dy$$

$$\Rightarrow dz = \frac{1}{k} dx + d\left(y \cdot \frac{k}{x}\right)$$

$$\Rightarrow z = \frac{x}{k} + \frac{yk}{x} + h$$

$$\Rightarrow (z+h) = \frac{x^2 + yk^2}{kx}$$

$$\Rightarrow kx(z+h) = x^2 + yk^2$$

$$\Rightarrow \boxed{kx(z+h) = k^2 y + x^2} \quad \text{--- (B)}$$

Now consider the curve from (B)

$$y = 0 \; ; \; x = k(z+h) \quad \text{--- (3)}$$

where h, k are independent parameters.

Now taking 't' as parameter in (3)

we get $z = t \; ; \; x = k(t+h) \; ; \; y = 0$ --- (4)

The intersection of (a) & (4) is

$$(t+b)^2 = 4[ak(t+h)]$$

$$\Rightarrow t^2 + (2b - 4ak)t + b^2 - 4akh = 0$$

This has equal roots if $(2b-4ak)^2-4(1)(b^2-4akh)=0$

$\Rightarrow b^2+4a^2k^2-4abk-b^2+4akh=0$

$\Rightarrow a^2k^2-abk+akh=0$

$\Rightarrow ak[ak-b+h]=0$

$\Rightarrow \boxed{b=h+ak} \quad (\because ak\neq 0)$

(a) $\equiv [z+(h+ak)]^2=4(ax+y)$ —— (5)

which is the one-parameter subsystem of (1)

(5) $\equiv z^2+(h+ak)^2+2z(h+ak)=4ax+4y$

$\Rightarrow z^2+h^2+a^2k^2+2hak+2hz+2kza-4ax-4y=0$

$\Rightarrow k^2a^2+(2hk+2zk-4x)a+z^2+2hz+h^2-4y=0$

This has equal roots

if $(2hk+2zk-4x)^2-4k^2((z+h)^2-4y)=0$

$\Rightarrow (hk+zk-2x)^2=k^2[(z+h)^2-4y)]$

$\Rightarrow h^2k^2+z^2k^2+4x^2+2hzk^2-4xzk-4xhk$
$=2hzk^2+k^2z^2+k^2h^2-4yk^2$

$\Rightarrow 4x^2-4xzk-4xhk=-4yk^2$

$\Rightarrow x^2+yk^2=xzk+xhk$

$\Rightarrow kx(z+h)=k^2y+x^2$

→ Show that the diff. eqn $2xz+q^2=x(xp+yq)$ has a complete integral $z+a^2x=axy+bx^2$ and deduce that $x(y+hx)^2=4(z-kx^2)$ is also a complete integral

→ Find the complete integral of diff. eqn $xp(1+q)=(y+z)q$ corresponding to the integral of Charpit's eqns which involving only q & x, and deduce that $(z+hx+k)^2=4hx(k-y)$

III The determination of surfaces which satisfy the PDE $F(x,y,z,p,q)=0$ —— ①
and which satisfy some other condition such as circumscribing a given surface.

→ Two surfaces are said to be circumscribe each other if they touch along curve.

→ Now we shall suppose that $f(x,y,z,a,b)=0$ —— ②
is a complete integral of ①.

Now we wish to find, by using ②, an integral surface of ①, which circumscribes the surface 'Σ' whose eqn. is $\psi(x,y,z)=0$ —— ③

If we have a surface E; $u(x,y,z)=0$ —— ④
of the required kind then it will be one of the three kinds:

(a) A particular case of the complete integral $f(x,y,z,a,b)=0$
obtained by giving particular values to a or b.

(b) A particular case of the general integral corresponding to ②
i.e, the envelope of a one-parameter subfamily of ②.

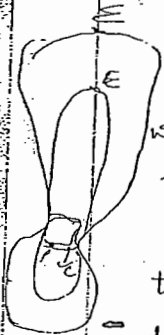
(c) The envelope of two-parameter system ②.

→ We now to find the surface ② which touch Σ and see if they provide a solution of the problem.

→ The surface ②, touches the surface ④ iff the eqns ②, ③ and $\frac{f_x}{\psi_x}=\frac{f_y}{\psi_y}=\frac{f_z}{\psi_z}$ —— ⑤ are consistent

Now eliminating x, y and z from these eqns, we get the relation b/w a & b of the form $\xi(a,b)=0$ —— (6)

This relation may be factorized into set of alternative eqns $b=\phi_1(a)$, $b=\phi_2(a)$, ------ —— (7)

each of which defines a subsystem of (2) whose members touch (3).

The points of contact lie on the surface whose eqn is obtained by eliminating a & b from the eqns (6) & (7).

The curve C is the intersection of this surface with Σ. Each of the relations (7) defines a subsystem whose envelope E touches Σ along C.

→ Show that the only integral surface of the eqn $2q(z-px-qy)=1+q^2$ which is circumscribed about the paraboloid $2x=y^2+z^2$ is the enveloping cylinder which touches it along its section by the plane $y+1=0$.

Soln: Given that $2q(z-px-qy)=1+q^2$ —— (1)

$$\Rightarrow z-px-qy=\frac{1+q^2}{2q}$$

$$\Rightarrow z=px+qy+\frac{1+q^2}{2q} \quad \text{—— (2)}$$

~~Clearly which is in clairaut's form~~

~~$z=px+qy+f(p,q)$~~

The required complete integral of (1) is

$$z=ax+by+\frac{1+b^2}{2b} \quad (\text{by putting } p=a, q=b)$$

Let $f(x,y,z,a,b) = z - ax - by - \frac{1+b^2}{2b} = 0$ —— (3)

and given that integral surface of (1) which circumscribes about the paraboloid $2x = y^2+z^2$

Let $\psi(x,y,z) = 2x - y^2 - z^2$ —— (4)

Now $\frac{f_x}{\psi_x} = \frac{f_y}{\psi_y} = \frac{f_z}{\psi_z}$

$\Rightarrow \frac{a}{2} = \frac{b}{-2y} = \frac{-1}{-2z}$ —— (5)

$\Rightarrow \frac{a}{2} = \frac{b}{-2y}$ & $\frac{a}{2} = \frac{1}{2z}$

$\Rightarrow \boxed{y = -\frac{b}{a}}$ & $\boxed{z = \frac{1}{a}}$ —— (6)

Now eliminating x b/w (3) & (4), we get

$2z = a(y^2+z^2) + 2by + \frac{2(b^2+1)}{2b}$

$\Rightarrow 2bz = aby^2 + abz^2 + 2b^2y + b^2+1$ —— (7)

and eliminating y & z from (7) by using (6) we get

$(b-a)(b^2+1) = 0$

$\Rightarrow \boxed{b=a}$ $(\because b^2+1 \neq 0)$

which defines a subsystem of (3) whose envelope is a surface of the required kind.

$\therefore$ The envelope of the subsystem

$[2(x+y)+1]a^2 - 2az + 1 = 0$ is

$4z^2 - 4[2(x+y)+1] \cdot 1 = 0$

$\Rightarrow z^2 = 2(x+y)+1$ —— (8)

Since the surface (4) touches the surface (8)

$2x - y^2 = 2(x+y)+1$

$\Rightarrow -y^2 = 2y+1$

$\Rightarrow y^2+2y+1 = 0$

$$\Rightarrow (y+1)^2 = 0$$

$$\Rightarrow y+1 = 0$$

→ Find the integral surface of the PDE $(y+zq)^2 = z(1+p^2+q^2)$ circumscribed about the surface $x^2-z^2 = 2y$

→ Show that the integral surface of the eqn $2y(1+p^2) = pq$ which is circumscribed about the cone $x^2+y^2 = y^2$ has eqn

$$z^2 = y^2(4y^2+4x+1).$$

Jacobi's Method

Working rule for solving PDE's with three (or) more than three independent variables:

Step 1: Suppose the given eqn with three independent variables is $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ —— ① in which the dependent variable z does not appear; x_1, x_2, x_3 are independent variables and $p_i = \frac{\partial z}{\partial x_i}$; $i = 1, 2, 3$.

Step 2: Write down the Jacobi's auxiliary eqns

$$\frac{dx_1}{-\partial f/\partial p_1} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_1}{\partial f/\partial x_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dp_3}{\partial f/\partial x_3}$$

Solving these eqns, we obtain two additional eqns

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \quad \text{——} ②$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \quad \text{——} ③$$

where a_1 & a_2 arbitrary constants.

Step 3: Verify that relations ② & ③ satisfy the condition

$$(F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0$$

$$\Rightarrow (F_1, F_2) = \sum_{r=1}^{3} \frac{\partial (F_1, F_2)}{\partial (x_r, p_r)} = 0 \quad \text{——} ④$$

If ④ is satisfied then solve ①, ② & ③ for p_1, p_2, p_3 in terms of x_1, x_2, x_3.

∴ Substitute these values in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

It gives the complete integral of the given eqn and containing three arbitrary constants.

Note: While solving a PDE with four independent variables,

Step 1: The given eqn is of the form

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0 \quad \text{---(1)}$$

Step 2: write down the Jacobi's auxiliary eqns

$$\frac{dx_1}{-\partial f/\partial p_1} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dx_4}{-\partial f/\partial p_4} = \frac{dp_1}{\partial f/\partial x_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dp_4}{\partial f/\partial x_4}$$

Solving these eqns, we obtain three additional eqns

$$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1 \quad \text{---(2)}$$

$$F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2 \quad \text{---(3)}$$

$$F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3 \quad \text{---(4)}$$

where a_1, a_2, a_3 are arbitrary constants

Step 3: verify the relations (2), (3), & (4) satisfy the following three conditions—

$$(F_1, F_2) = \sum_{r=1}^{4} \frac{\partial(F_1, F_2)}{\partial(x_r, p_r)} = 0 \quad \text{---(5)}$$

$$(F_2, F_3) = \sum_{r=1}^{4} \frac{\partial(F_2, F_3)}{\partial(x_r, p_r)} = 0 \quad \text{---(6)}$$

and $(F_3, F_1) = \sum_{r=1}^{4} \frac{\partial(F_3, F_1)}{\partial(x_r, p_r)} = 0$ ---(7)

If (5), (6) & (7) are satisfied then solve (1), (2), (3) & (4) for p_1, p_2, p_3 & p_4 in terms of x_1, x_2, x_3 & x_4 and substitute these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$

which gives the complete integral of (1) and containing four arbitrary constants.

97 → Find a complete integral of $p_1^3 + p_2^2 + p_3 = 1$

Sol^n: Let the given eqn be

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0 \quad \text{---(1)}$$

Now Jacobi's A.E's are

$$\frac{dx_1}{-\partial f/\partial p_1} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_1}{\partial f/\partial x_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dp_3}{\partial f/\partial x_3}$$

$$\Rightarrow \frac{dx_1}{-3p_1^2} = \frac{dx_2}{-2p_2} = \frac{dx_3}{-1} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}.$$

from the 4th & 5th fractions, we get

$\therefore dp_1 = 0$ and $dp_2 = 0$

$\Rightarrow p_1 = a_1$ and $p_2 = a_2$

Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 - a_1 = 0$ —— (2)

$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 - a_2 = 0$ —— (3)

Now $(F_1, F_2) = \sum_{r=1}^{3} \frac{\partial(F_1, F_2)}{\partial(x_r, p_r)}$

$$= \frac{\partial(F_1,F_2)}{\partial(x_1,p_1)} + \frac{\partial(F_1,F_2)}{\partial(x_2,p_2)} + \frac{\partial(F_1,F_2)}{\partial(x_3,p_3)}$$

$$= \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

$= 0$

$\therefore (F_1, F_2) = 0$

$\therefore$ The eqns (2) & (3) taken as additional eqns.

$\therefore$ solving (1), (2) & (3) for p_1, p_2 & p_3.

we have $p_1 = a_1$, $p_2 = a_2$ & $p_3 = 1 - a_1^3 - a_2^2$.

$\therefore$ putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$

we have
$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3$$

Integrating, we get

$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3$$

where a_1, a_2, a_3 are arbitrary constants.
which is the required complete integral.

1998 → Find a complete integral of $2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0$

→ find a complete integral of $p_1 p_2 p_3 = z^3 x_1 x_2 x_3$

i.e, $\frac{\partial z}{\partial x_1} \cdot \frac{\partial z}{\partial x_2} \cdot \frac{\partial z}{\partial x_3} = z^3 x_1 x_2 x_3$

Soln: $\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = x_1 x_2 x_3$ ——— (1)

Taking $\frac{1}{z}dz = dZ$

$\Rightarrow \boxed{\log z = Z}$

$\therefore$ (1) $\equiv \left(\frac{\partial Z}{\partial x_1}\right)\left(\frac{\partial Z}{\partial x_2}\right)\left(\frac{\partial Z}{\partial x_3}\right) = x_1 x_2 x_3$

Let $f(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 P_2 P_3 - x_1 x_2 x_3 = 0$ ——— (2)

where $P_1 = \frac{\partial Z}{\partial x_1}$, $P_2 = \frac{\partial Z}{\partial x_2}$, $P_3 = \frac{\partial Z}{\partial x_3}$

Now the Jacobi's auxiliary eqns are

$$\frac{dx_1}{-P_2P_3} = \frac{dx_2}{-P_1P_3} = \frac{dx_3}{-P_1P_2} = \frac{dP_1}{-x_2x_3} = \frac{dP_2}{-x_1x_3} = \frac{dP_3}{-x_1x_2} \text{ ——— (3)}$$

(3) $\equiv P_2 P_3 = \frac{x_1 x_2 x_3}{P_1}$

$\therefore$ first and fourth fractions of (3) give

$$\frac{dx_1}{-\frac{x_1x_2x_3}{P_1}} = \frac{dP_1}{-x_2x_3} \Rightarrow \frac{dP_1}{P_1} = \frac{dx_1}{x_1}$$

Integrating, we get

$\boxed{P_1 = a_1 x_1}$

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 - a_1 x_1$ ——— (4)

Similarly we have $F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 - a_2 x_2$ ——— (5)

$$(F_1, F_2) = \sum_{r=1}^{3} \frac{\partial(F_1, F_2)}{\partial(x_r, P_r)} = \frac{\partial(F_1, F_2)}{\partial(x_1, P_1)} + \frac{\partial(F_1, F_2)}{\partial(x_2, P_2)} + \frac{\partial(F_1, F_2)}{\partial(x_3, P_3)}$$

$$= \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial P_1} - \frac{\partial F_1}{\partial P_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial P_2} - \frac{\partial F_1}{\partial P_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial P_3} - \frac{\partial F_1}{\partial P_3}\frac{\partial F_2}{\partial x_3}$$

$= 0$

$\therefore$ The eqns (4) & (5) taken as additional eqns.

Solving (2), (4) & (5), we get $P_1 = a_1 x_1$, $P_2 = a_2 x_2$,

$P_3 = \frac{x_3}{a_1 a_2}$

Putting these values in $dZ = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$

$dZ = a_1 x_1 dx_1 + a_2 x_2 dx_2 + \frac{x_3}{a_1 a_2} dx_3$

Integrating, we get

$Z = \frac{1}{2} a_1 x_1^2 + \frac{1}{2} a_2 x_2^2 + \frac{1}{2 a_1 a_2} x_3^2 + a_3$

Taking $Z = \log z$

$\therefore 2 \log z = a_1 x_1^2 + a_2 x_2^2 + \frac{x_3^2}{a_1 a_2} + a_3$, which is the required complete integral.

Cauchy's Method of Characteristics:

For solving non-linear differential eqns:

Working rule:

Let us consider the non-linear PDE

$f(x, y, z, p, q) = 0$ ——— ①

Suppose we wish to find the solution of ① which passes through a given curve whose parametric eqns are

$x = f_1(\lambda), \; y = f_2(\lambda), \; z = f_3(\lambda)$ ——— ②

where λ is a parameter.

Then in the solution

$$\left.\begin{aligned} x &= x(p_0, q_0, x_0, y_0, z_0, t_0, t) \\ y &= y(p_0, q_0, x_0, y_0, z_0, t_0, t) \text{ and} \\ z &= z(p_0, q_0, x_0, y_0, z_0, t_0, t) \end{aligned}\right\} \text{——— ③}$$

of the characteristics eqns of ① are

$x'(t) = f_p, \; y'(t) = f_q, \; z'(t) = pf_p + qf_q$

$p'(t) = -f_x - pf_z$ and $q'(t) = -f_y - qf_z$ ——— ④

where $x'(t) = \frac{dx}{dt}$ etc

and $f_p = \frac{\partial f}{\partial p}$ etc.

We shall assume that

$x_0 = f_1(\lambda), \; y_0 = f_2(\lambda), \; z_0 = f_3(\lambda)$ as the initial values of x, y, z respectively. Then the corresponding initial values of p_0, q_0 are determined by the following relations

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$ &

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

If these values of x_0, y_0, z_0, p_0, q_0 and the

appropriate value of 'to' substitute in the eqn ③.

we find that x, y, z can be expressed interms of the two parameters t & λ of the form

$x = \phi_1(t,\lambda)$, $y = \phi_2(t,\lambda)$ & $z = \phi_3(t,\lambda)$ —— ④

which are known as characteristic strips of ①.

Finally by eliminating λ & t from ④, we get the relation of the form $\psi(x,y,z)=0$ which is the required integral surface of ① passing through the given curve ②.

2002 → Find the solution of the eqn $z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$ which passes through the x-axis.

Soln: Given that $z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$

Let $f(x,y,z,p,q) = \frac{1}{2}(p^2+q^2) + (p-x)(q-y) - z$ —— ①

we are to find the integral surface of ① which passes through x-axis whose parametric eqns are $x = \lambda$, $y = 0$, $z = 0$ where λ is the parameter.

i.e., $x = f_1(\lambda) = \lambda$, $y = f_2(\lambda) = 0$, $z = f_3(\lambda) = 0$ —— ②

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as $x_0 = f_1(\lambda) = \lambda$; $y_0 = f_2(\lambda) = 0$, $z_0 = f_3(\lambda) = 0$.

Now we find the initial values p_0 & q_0 by the following relations

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$ &

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

i.e. $f(x_0, y_0, z_0, p_0, q_0) = 0$

$\Rightarrow 0 = p_0(1) + q_0(0)$ &

$\frac{1}{2}(p_0^2 + q_0^2) + (p_0 - x_0)(q_0 - y_0) - z_0 = 0$

$\Rightarrow \boxed{p_0 = 0}$ & $\frac{1}{2}q_0^2 - x_0(q_0 - y_0) - z_0 = 0$ $(\because p_0 = 0)$

$\Rightarrow \frac{1}{2}q_0^2 - \lambda(q_0 - 0) - 0 = 0$

$\Rightarrow q_0(\frac{1}{2}q_0 - \lambda) = 0$

$\Rightarrow \frac{1}{2}q_0 = \lambda$ $(\because q_0 \neq 0)$

$\Rightarrow \boxed{q_0 = 2\lambda}$

$\therefore x_0 = \lambda, y_0 = 0, z_0 = 0, p_0 = 0$, & $q_0 = 2\lambda$ & $t = t_0$ ——— (3)

Now the characteristic eqns of (1) are

$x'(t) = \frac{\partial f}{\partial p} = p + (q - y)$ ——— (4)

$y'(t) = \frac{\partial f}{\partial q} = q + (p - x)$ ——— (5)

$z'(t) = p[p + q - y] + q[q + p - x]$ ——— (6)

$p'(t) = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z}$

$= (q - y) - p(-1)$

$= q - y + p$ ——— (7)

$q'(t) = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z}$

$= (p - x) - q(-1) = p - x + q$ ——— (8)

from (4) & (7), we have $x'(t) = p'(t)$

$\Rightarrow \frac{dx}{dt} = \frac{dp}{dt}$

$\Rightarrow dx = dp$

$\Rightarrow \boxed{x = p + c_1}$ ——— (9)

From (5) & (8), we get

$$y'(t) = q'(t)$$

$$\Rightarrow dy = dq$$

$$\Rightarrow \boxed{y = q + c_2} \quad \text{---} (10)$$

Using the initial values in (9) & (10)

(9) ≡ $\lambda = 0 + c_1$; (10) ≡ $0 = 2\lambda + c_2$

$\Rightarrow \boxed{c_1 = \lambda}$ $\qquad \Rightarrow \boxed{c_2 = -2\lambda}$

∴ from (9) & (10), we have

$$\boxed{x = p + \lambda} \quad \& \quad \boxed{y = q - 2\lambda} \quad \text{---} (11)$$

from (4), (7) & (8), we get

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = p + q - x$$

$$\Rightarrow \frac{d}{dt}(p + q - x) = p + q - x$$

$$\Rightarrow \frac{d(p+q-x)}{p+q-x} = dt$$

$$\Rightarrow \log(p+q-x) = t + \log c_3$$

$$\Rightarrow \boxed{p + q - x = c_3 e^t} \quad \text{---} (12)$$

from (5), (7) & (8), we get

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = p + q - y$$

$$\Rightarrow \frac{d(p+q-y)}{p+q-y} = dt$$

$$\Rightarrow \boxed{p + q - y = c_4 e^t} \quad \text{---} (13)$$

Using the initial values in (12) & (13), we get

$$p_0 + q_0 - x_0 = c_3 e^{t_0} \quad \& \quad p_0 + q_0 - y_0 = c_4 e^{t_0}$$

Taking $t = t_0 = 0$.

$$\Rightarrow p_0 + q_0 - x_0 = c_3 \quad \& \quad 0 + 2\lambda - 0 = c_4$$

$\Rightarrow \boxed{\lambda = c_3}$ & $\boxed{c_4 = 2\lambda}$

from (12) & (13), we get

$p+q-x = \lambda e^t$ — (14) & $p+q-y = 2\lambda e^t$ — (15)

now from (9), (14) & (15), we get

$p+q-(p+\lambda) = \lambda e^t$; $p+q-(q-2\lambda) = 2\lambda e^t$

$\Rightarrow q-\lambda = \lambda e^t$ $\quad \boxed{p = 2\lambda(e^t-1)}$

$\Rightarrow \boxed{q = \lambda(1+e^t)}$

$\therefore$ from (11), we have

$x = \lambda(2e^t-1)$ — (16) & $y = \lambda(e^t-1)$ — (17)

(6) $\equiv \dfrac{dz}{dt} = p[p+q-y] + q[p+q-x]$

$= 2\lambda(e^t-1)(2\lambda e^t) + \lambda(1+e^t)(\lambda e^t)$ (from (14) & (15))

$= 5\lambda^2 e^{2t} - 3\lambda^2 e^t$

$dz = \lambda^2(5e^{2t}-3e^t)\,dt$

$z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t\right) + c_5$

By using the initial values.

$z_0 = \lambda^2\left(\frac{5}{2}e^{2t_0} - 3e^{t_0}\right) + c_5$

$\Rightarrow 0 = \lambda^2\left(\frac{5}{2} - 3\right) + c_5$

$\Rightarrow \boxed{c_5 = \frac{\lambda^2}{2}}$

$\boxed{z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t\right) + \frac{\lambda^2}{2}}$ — (18)

The required characteristic strips of (1) are given by

$x = \lambda(2e^t-1)$, $y = \lambda(e^t-1)$ & $z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t\right) + \frac{\lambda^2}{2}$ — (19)

Now eliminating t & λ from (19)

Now solving (i) & (ii) from (19) for λ & e^t, we get

$$x = 2\lambda\left(\frac{y+\lambda}{\lambda}\right) - \lambda \quad \left(\because \text{from (ii) } e^t = \frac{y+\lambda}{\lambda}\right)$$

$$\Rightarrow x = 2y + 2\lambda - \lambda$$

$$\Rightarrow x = 2y + \lambda$$

$$\Rightarrow \boxed{\lambda = x - 2y}$$

(ii) $\equiv$ $y = (x-2y)(e^t - 1)$

$$\Rightarrow e^t - 1 = \frac{y}{x-2y}$$

$$\Rightarrow \boxed{e^t = \frac{y}{x-2y} + 1} \Rightarrow \boxed{e^t = \frac{x-y}{x-2y}}$$

(iii) $\equiv$ $z = (x-2y)^2\left\{\left[\frac{5}{2}\left(\frac{x-y}{x-2y}\right)^2 - 3\left(\frac{x-y}{x-2y}\right)\right] + \frac{1}{2}\right\}$

which is the required solution of (1) passing through the given curve.

1999 Find characteristics of the eqn $pq = z$ and the integral surface which passes through the parabola $x = 0,\ y^2 = z$.

Sol: Given that $pq = z$

$$\Rightarrow f(x, y, z, p, q) = pq - z = 0 \quad \text{--- (1)}$$

Now we are to find the integral surface of (1) which is passing through the parabola

$x = 0,\ y^2 = z$

whose parametric eqns are

$x = 0,\ y = \lambda$ & $z = \lambda^2$

i.e, $x = f_1(\lambda),\ y = f_2(\lambda)$ & $z = f_3(\lambda)$ --- (2)

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Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = f_1(\lambda) = 0, \quad y_0 = f_2(\lambda) = \lambda, \quad z_0 = f_3(\lambda) = \lambda^2$

Now we find the initial values p_0 & q_0 by the following relations

$$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda) \quad \&$$

$$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$$

i.e, $f(0, \lambda, \lambda^2, p_0, q_0) = 0$

$\Rightarrow 2\lambda = p_0(0) + q_0(1)$ & $p_0 q_0 - \lambda^2 = 0$

$\Rightarrow 2\lambda = q_0$ $\qquad \Rightarrow p_0(2\lambda) - \lambda^2 = 0$

$\Rightarrow \boxed{q_0 = 2\lambda}$ $\qquad \Rightarrow p_0 = \frac{\lambda}{2}$

$\therefore x_0 = 0, \ y_0 = \lambda, \ z_0 = \lambda^2, \ q_0 = 2\lambda, \ p_0 = \lambda/2$ ——— (3)

Now the characteristic eqns of (1) are

$x'(t) = \frac{\partial f}{\partial p} = q$ ——— (4)

$y'(t) = \frac{\partial f}{\partial q} = p$ ——— (5)

$z'(t) = pq + pq = 2pq$ ——— (6)

$p'(t) = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -p(-1)$
$= p$ ——— (7)

$q'(t) = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z}$
$= -q(-1) = q$ ——— (8)

Now from (4) & (8), we have

$x'(t) = q'(t)$

$\Rightarrow dx = dq$

$\Rightarrow \boxed{x = q + c_1}$ ——— (9)

from (5) & (7), we have

$-y'(t) = p'(t) \Rightarrow dy = dp$

$\Rightarrow \boxed{y = p + c_2}$ —— (10)

Now using the initial values in (9) & (10), we get $x_0 = q_0 + c_1$ & $y_0 = p_0 + c_2$

$\Rightarrow 0 = 2\lambda + c_1$ & $\lambda = \frac{\lambda}{2} + c_2$

$\Rightarrow \boxed{c_1 = -2\lambda}$ $\Rightarrow \boxed{c_2 = \frac{\lambda}{2}}$

$\therefore$ from (9) & (10), we have

$\boxed{x = q - 2\lambda}$ —— (11) $\boxed{y = p + \frac{\lambda}{2}}$ —— (12)

(7) $\equiv \frac{dp}{dt} = p$

$\Rightarrow \frac{dp}{p} = dt \Rightarrow \log p = t + \log c_3$

$\Rightarrow \boxed{p = c_3 e^t}$ —— (13)

(8) $\equiv q'(t) = q \Rightarrow \frac{dq}{q} = dt$

$\Rightarrow \log q = t + \log c_4$

$\Rightarrow \boxed{q = c_4 e^t}$ —— (14)

Using the initial values in (13) & (14), we get

$p_0 = c_3 e^{t_0}$ & $q_0 = c_4 e^{t_0}$

$\Rightarrow \frac{\lambda}{2} = c_3$ & $2\lambda = c_4$ $(\because t_0 = 0)$

$\therefore$ from (13) & (14), we have

$p = \frac{\lambda}{2} e^t$ & $q = 2\lambda e^t$ —— (15)

(11) $\equiv x = 2\lambda e^t - 2\lambda \Rightarrow x = 2\lambda(1 - e^t)$ —— (16)

(12) $\equiv y = \frac{\lambda}{2} e^t + \frac{\lambda}{2} \Rightarrow y = \frac{\lambda}{2}(e^t + 1)$ —— (17)

(6) $\equiv z'(t) = 2pq$

$\Rightarrow \frac{dz}{dt} = 2\left(e^t \frac{\lambda}{2}\right)(2\lambda e^t)$

$\Rightarrow dz = 2\lambda^2 e^{2t} dt$

$\Rightarrow z = \lambda^2 e^{2t} + c_5$ —— (18)

Using the initial values in (18), we get

$z_0 = \lambda^2 e^{2t_0} + c_5$

$\Rightarrow \lambda^2 = \lambda^2 (1) + c_5$

$\Rightarrow \boxed{c_5 = 0}$

$\therefore$ (18) $\equiv \boxed{z = \lambda^2 e^{2t}}$

$\therefore$ The required characteristics of (1) are given by

$x = 2\lambda(e^t - 1)$ —— (19), $y = \frac{\lambda}{2}(e^t + 1)$ —— (20), $z = \lambda^2 e^{2t}$ —— (21)

Now eliminating e^t & λ from (19), (20) & (21)

(19) $\equiv x = \frac{4y}{e^t + 1}(e^t - 1)$

$\Rightarrow x e^t + x = 4y e^t - 4y$

$\Rightarrow e^t(x - 4y) = -(x + 4y)$

$\Rightarrow \boxed{e^t = \frac{-(x+4y)}{x-4y}}$

(20) $\equiv \lambda = \frac{2y}{e^t + 1} = \frac{2y}{\frac{-(x+4y)}{x-4y} + 1} = \frac{2y(x-4y)}{-8y}$

$\boxed{\lambda = \frac{x - 4y}{-4}}$

$\therefore z = \left(\frac{x-4y}{-4}\right)^2 \frac{(x+4y)^2}{(x-4y)^2}$

$\boxed{z = \frac{(x+4y)^2}{16}}$

which is the required integral surface of (1).

2001) P.T for the eqn $z+px+qy-1-pqx^2y^2=0$ the characteristic strips are given by

$$x=\frac{1}{B+Ce^{t}},\quad y=\frac{1}{A+De^{t}},\quad z=E-(AC+BD)e^{-t}$$

$$p=A(B+Ce^{-t})^2,\quad q=B(A+Be^{-t})^2$$

where A, B, C, D and E are arbitrary constants. Hence find the integral surface which passes through the line $z=0$, $x=y$.

Soln: Given that $z+px+qy-1-pqx^2y^2=0$

Let $f(x,y,z,p,q) = z+px+qy-1-pqx^2y^2=0$ —— ①

we are to find integral surface of ① which passes through the line $x=y$, $z=0$ — whose parametric eqns are

$x=\lambda$, $y=\lambda$ & $z=0$

i.e, $x=f_1(\lambda)=\lambda$; $y=f_2(\lambda)=\lambda$ & $z=f_3(\lambda)=0$ —— ②

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0=f_1(\lambda)=\lambda$, $y_0=f_2(\lambda)=\lambda$, $z_0=f_3(\lambda)=0$

Now we find the initial values p_0 & q_0 by the relations

$f_3'(\lambda)=p_0 f_1'(\lambda)+q_0 f_2'(\lambda)$ &

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0)=0$

i.e, $f(\lambda,\lambda,0,p_0,q_0)=0$

$\Rightarrow$ $0=p_0(1)+q_0(1)$ & $0+p_0(\lambda)+q_0(\lambda)-1-p_0q_0\lambda^4=0$

$\Rightarrow$ $\boxed{p_0+q_0=0}$ —(i) $\Rightarrow$ $\lambda(p_0+q_0)-1-p_0q_0\lambda^4=0$

$\Rightarrow$ $\lambda(0)-p_0q_0\lambda^4=0$

$\Rightarrow$ $\boxed{p_0q_0=-\frac{1}{\lambda^4}}$

Now $(p_0-q_0)^2 = (p_0+q_0)^2 - 4p_0q_0$

$= 0 + \frac{4}{\lambda^4}$

$p_0 - q_0 = \frac{2}{\lambda^2}$ —(ii)

from (i) & (ii) $2p_0 = \frac{2}{\lambda^2}$

$\Rightarrow p_0 = \frac{1}{\lambda^2}$ & $q_0 = -\frac{1}{\lambda^2}$

$\therefore x_0 = \lambda,\ y_0 = \lambda,\ z_0 = 0;\ p_0 = \frac{1}{\lambda^2}$ & $q_0 = -\frac{1}{\lambda^2}$ —(3)

Now the characteristic eqns of (1) are

$x'(t) = f_p = x - qx^2y^2$ —(4)

$y'(t) = f_q = y - px^2y^2$ —(5)

$z'(t) = p[x - qx^2y^2] + q[y - px^2y^2]$

$= px + qy - 2pqx^2y^2$ —(6)

$p'(t) = -[p - 2pqxy^2] - p(1)$

$= -2p[1 - qxy^2]$ —(7)

$q'(t) = -[q - 2pqx^2y] - q(1)$

$= -2q[1 - px^2y]$ —(8)

from (4) & (7), we have

$x'(t) = x\left(\frac{p'(t)}{-2p}\right)$

$\Rightarrow -\frac{2}{x}dx = \frac{1}{p}dp$

$\Rightarrow -2\log x = \log p + \log c_1$

$\Rightarrow x^{-2} = pc_1$ —(9)

from (5) & (8), we have

$y'(t) = y\left(\frac{-q'(t)}{2q}\right)$

$\Rightarrow -\frac{2}{y}dy = \frac{1}{q}dq.$

$\Rightarrow -2\log y = \log q + \log c_2$

$\Rightarrow \boxed{y^{-2} = q c_2}$ —— (10)

Using the initial values in (9) & (10), we get

$x_0^{-2} = p_0 c_1$ & $y_0^{-2} = q_0 c_2$

$\Rightarrow \lambda^{-2} = \frac{1}{\lambda^2} c_1$ & $\lambda^{-2} = -\frac{1}{\lambda^2} c_2$

$\Rightarrow \boxed{c_1 = 1}$ & $\boxed{c_2 = -1}$

$\therefore$ from (9) & (10), we have

$\boxed{\frac{1}{x^2} = p}$ & $\boxed{-\frac{1}{y^2} = q}$ —— (11)

Continuing in this way.

2000 Find the characteristic strips of the eqn $xp + yq - pq = 0$ and then find the eqn of integral surface through curve $z = \frac{x}{2}, y = 0$

→ Write down and integrate completely, the equations for the characteristics of $(1+q^2)z = px$. Expressing x, y, z and p in terms of ϕ, where $q = \tan\phi$ and determine the integral surface which passes through parabola $x^2 = 2z, y = 0$

→ Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$.

→ Integrate the eqns of the characteristics of the eqn $p^2 + q^2 = 4z$.

Expressing x, y, z and p in terms of q and then find the solutions of this equation which reduce to $z = x^2 + 1, y = 0$

Set-III

Linear partial Differential eqns with constant coefficients:

The general linear partial differential eqn of an order higher than the first:

A PDE in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients all being constants (or) fns of x & y, is called a linear PDE.

The general form of such an eqn can be written in the form

$$\left(\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + A_n\frac{\partial^n z}{\partial y^n}\right)$$

$$+\left(B_0\frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1\frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + B_2\frac{\partial^{n-1} z}{\partial x^{n-3}\partial y^2} + \cdots + B_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)$$

$$+ \cdots + \left(M_0\frac{\partial z}{\partial x} + M_1\frac{\partial z}{\partial y}\right) + N_0 z = f(x,y) \quad \text{--- (1)}$$

where the coefficients $A_1, A_2, \ldots A_n$; $B_0, B_1, \ldots B_{n-1}$, ---- M_0, M_1, N_0 are constants (or) functions of x & y.

If the coefficients of various terms are constants then (1) is called a linear PDE with constant coefficients.

If all the derivatives appearing in (1) are of the same order then the resulting eqn is called a linear homogeneous PDE with constant coefficients and it is of the form

$$\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{--- (2)}$$

where $A_0, A_1, \ldots A_n$ are constants.

Denoting the operators $\frac{\partial}{\partial x}$ by D ; $\partial/\partial y$ by D',

$\therefore$ (2) $\Rightarrow \left[D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + A_3 D^{n-3} D'^3 + \cdots \cdots + A_n D'^n\right] z = f(x,y)$

$\Rightarrow F(D, D') z = f(x,y)$ —— (3)

where $F(D,D') = D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots \cdots + A_n D'^n$

Note:– $F(D, D')$ is a homogeneous function in D, D' of degree 'n'.

Solution of a linear homogeneous partial Differential eqn with constant coefficients:

→ If u is the complementary function (C.F) and z' a particular integral (P.I) of a linear PDE $F(D,D')z = f(x,y)$ then $u + z'$ is a g.s of the linear PDE.

→ If $u_1, u_2, \cdots, u_n$ are solns of the homogeneous linear PDE $F(D,D')z = 0$ then $\sum_{r=1}^{n} c_r u_r$ is also soln, where $c_1, c_2, \cdots c_n$ are arbitrary constants.

Determination of the C.F of the linear homogeneous PDE with constant coefficients $F(D,D')z = f(x,y)$:

Let $F(D,D')z = f(x,y)$ be the given linear homo. PDE with constant coeff.

then $\left[D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right] z = f(x,y)$ —— (1)

where $A_1, A_2, \cdots A_n$ are constants.

(3)

The complimentary function (C.F) of (1) is the g.s of $F(D,D')z = 0$

i.e. $\left[D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n\right] z = 0$ → (2)

$\Rightarrow \left[(D - m_1 D')(D - m_2 D')(D - m_3 D') \cdots (D - m_n D')\right] z = 0$ → (3)

where $m_1, m_2, \ldots, m_n$ are some constants.

The soln of any one of the eqns

$(D - m_1 D')z = 0,\ (D - m_2 D')z = 0, \ldots, (D - m_n D')z = 0$ → (4)

is also a soln of (3).

We now show that the general soln of $(D - mD')z = 0$ is $z = \phi(y + mx)$, where ϕ is an arbitrary fn.

now $(D - mD')z = 0 \Rightarrow \frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = 0$

$\Rightarrow p + (-m)q = 0$ → (5)

clearly which is in Lagrange's form

$Pp + Qq = R$

∴ the Lagrange's auxiliary eqns of (5) are

$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}$ → (6)

now taking first two fractions of (6)

$\frac{dx}{1} = \frac{dy}{-m} \Rightarrow dy + m\,dx = 0$

integrating, we get

$\Rightarrow y + mx = c_1$ → (7)

now taking third fraction of (6)

$dz = 0$

$\Rightarrow \boxed{z = c_2}$

(4)

∴ from (7) & (8), the g.s of (5) is $z = \phi(y+mx)$

where ϕ is arbitrary function.

We assume that a soln. of (2) is of the form $z = \phi(y+mx)$ — (9)

where ϕ is arbitrary fn & m is const.

Now from (9),

$$Dz = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\phi(y+mx)] = m\phi'(y+mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2 \phi''(y+mx)$$

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n \phi^{(n)}(y+mx)$$

and

$$D' z = \frac{\partial z}{\partial y} = \phi'(y+mx)$$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = \phi''(y+mx)$$

$$D'^n z = \frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y+mx)$$

Also, in general, $D^r D'^s z = \frac{\partial^{r+s} z}{\partial x^r \partial y^s} = m^r \phi^{(r+s)}(y+mx)$

(2) ≡ $(m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n)\phi^{(n)}(y+mx) = 0$

This is true if m is a root of the eqn. $m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0$ — (10)

The eqn (10) is called the auxiliary eqn (A.E) obtained by putting $D = m$, $D' = 1$ in (2) $F(D,D') = 0$

(9)

In general, the eqn (10) can give 'n' roots, say $m_1, m_2, \ldots, m_n$.

Each value of 'm' will give a soln of (2).

→ If all the roots of the auxiliary eqn (10) are distinct, the g.s of (2) i.e. the C.F of (1) is

$$z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \cdots + \phi_n(y+m_nx)$$ —— (11)

$$\text{i.e. } z = \sum_{r=1}^{n} \phi_r(y+m_rx) \; ; \; r = 1, 2, \ldots, n.$$

Case of equal roots:

If the auxiliary eqn (10) has two equal roots i.e. $m_1 = m_2$

i.e. $m = m_1, m_1$

Now consider the eqn

$$(D - mD')(D - mD')z = 0$$ —— (12) (from (8))

putting $(D - mD')z = u$ —— (13) in (12) we get

$$(D - mD')u = 0$$

∴ the soln of which is $u = \phi(y+mx)$.

$$\therefore (D - mD')z = \phi(y+mx) \quad \text{(from (13))}$$

$$p - mq = \phi(y+mx)$$ —— (14)

Lagrange's auxiliary eqns of (14) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y+mx)}$$ —— (15)

Taking the first two fractions of (15) we get

$dy = -m\,dx \Rightarrow y + mx = a$ —— (16) where a is arbitrary const.

Taking first & last fractions (6) & (15), we get

$$\frac{dx}{1} = \frac{dz}{\phi(a)} \quad (\text{from } (15))$$

$\Rightarrow dz = \phi(a)\, dx$

$\Rightarrow z = x\,\phi(a) + b$ where b is arbitrary

$$\Rightarrow \boxed{z = x\,\phi(y+mx) + b}$$

Since b is arbitrary,

taking $b = \phi_1(a)$

$\therefore$ The soln of (10) is

$$z = x\,\phi(y+mx) + \phi_1(a)$$

$$\boxed{z = x\,\phi(y+mx) + \phi_1(y+mx)}$$

where ϕ & ϕ_1 are arbitrary fns.

Note: Proceeding in the same way,

If the auxiliary eqn (10) has r equal roots then the C.F of (1) is

$$z = \phi_1(y+mx) + x\,\phi_2(y+mx) + x^2\,\phi_3(y+mx) + \cdots$$
$$\cdots + x^{r-1}\phi_r(y+mx).$$

Working rule for finding C.F:

Step 1: Write down the given eqn in standard form

$$(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n) z = f(x,y) \quad (1)$$

Step (2): Replacing D by m and D' by 1 in the coefficient of z, we obtain the A.E for (1)

as $m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0$, — (2)

Step (I): solve (2) for 'm'.

Some cases will arise:

case (i): Let $m = m_1, m_2, \ldots m_n$ (distinct roots)

then C.F. of (1) $= \phi_1(y+m_1x) + \phi_2(y+m_2x) + \ldots + \phi_n(y+m_nx)$.

where $\phi_1, \phi_2, \ldots \phi_n$ are arbitrary fns.

case (ii): Let $m = m'$ (repeated n times)

then C.F. of (1) $=$
$\phi_1(y+m'x) + x\phi_2(y+m'x) + x^2\phi_3(y+m'x) + \ldots + x^{n-1}\phi_n(y+m'x)$.

case (iii): corresponding to a non-repeated factor D on LHS of (1), the part of C.F. is taken as $\phi(y)$.

case (iv): corresponding to a repeated factor D^m on LHS of (1), the part of C.F. is taken as $\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \ldots + x^{m-1}\phi_m(y)$.

case (v): corresponding to a non-repeated factor D' on LHS of (1), the part of C.F. is taken as $\phi(x)$.

case (vi): corresponding to a repeated factor D'^m on LHS of (1), the part of C.F. is taken as
$\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \ldots + y^{m-1}\phi_m(x)$.

Alternative working rule for finding C.F

Let the given diff eqn be $F(D, D')z = f(x,y)$.

Factorise $F(D, D')$ into linear factors of the form $(bD - aD')$.

Then we use the following results:

(i) corresponding to each non-repeated factor $(bD - aD')$, the part of C.F is taken as $\phi(by + ax)$.

(ii) corresponding to a repeated factor $(bD - aD')^m$, the part of C.F is taken as $\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + \cdots + x^{m-1}\phi_m(by + ax)$.

(iii) corresponding to a non-repeated factor D, part of C.F is taken as $\phi(y)$

(iv) corresponding to a repeated factor D^m, the part C.F is taken as
$\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \cdots + x^{m-1}\phi_m(y)$

(v) correspond to a non-repeated factor D', the part of C.F is taken as $\phi(x)$.

(vi) corresponding to a repeated factor D'^m, the part of C.F is taken as
$\phi_1(x) + y\phi_2(x) + \cdots + y^{m-1}\phi_m(x)$.

NOTE:– $p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x \partial y}$
$t = \frac{\partial^2 z}{\partial y^2}$.

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problems

→ solve $2r+5s+2t=0$.

Sol: Given that $2r+5s+2t=0$ —(1)

W.K.T $r=\frac{\partial^2 z}{\partial x^2} = D^2 z$; $s=\frac{\partial^2 z}{\partial x \partial y} = DD'z$ & $t=\frac{\partial^2 z}{\partial y^2} = D'^2 z$

∴ (1) ≡

$$[2D^2+5DD'+2D'^2]z=0. \quad \text{—(2)}$$

A.E of (2) is

$2m^2+5m+2=0$ $(\because D=m, D'=1)$

$\Rightarrow (2m+1)(m+2)=0$

$\Rightarrow m=-\frac{1}{2}, -2$

∴ the g.s of (1) is

$$z=\phi_1(y-\tfrac{1}{2}x)+\phi_2(y-2x).$$

where ϕ_1 & ϕ_2 arbitrary fns.

→ solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$. —(1)

$(D^2+D'^2)z=0$ —(2)

A.E of (2) is $m^2+1=0$.

$\Rightarrow m^2=-1$

$\Rightarrow \boxed{m=\pm i}$

∴ the g.s of (1) is

$$z=\phi_1(y+ix)+\phi_2(y-ix).$$

→ solve $r+t+2s=0$.

Given that $r+t+2s=0$ —(1)

$\Rightarrow \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+2\frac{\partial^2 z}{\partial x \partial y}=0$.

$\Rightarrow (D^2+2DD'+D'^2)z=0$

Particular Integral :

Let us consider an equation

$$F(D,D')z = f(x,y) \quad \text{---} (1)$$

then P.I of (1) is denoted by

$$\frac{1}{F(D,D')} f(x,y).$$

Note: (1) $\frac{1}{D}$ means integration partially w.r.t 'x'.

$\frac{1}{D'}$ means integration partially w.r.t 'y'.

(2) $D[\phi(ax+by)] = \frac{\partial}{\partial x}[\phi(ax+by)]$

$= a\phi'(ax+by).$

$D'[\phi(ax+by)] = \frac{\partial}{\partial y}[\phi(ax+by)]$

$= b[\phi'(ax+by)]$

In general,

$D^r[\phi(ax+by)] = a^r \phi^{(r)}(ax+by)$

$D'^s[\phi(ax+by)] = b^s \phi^{(s)}(ax+by)$

and $D^r D'^s[\phi(ax+by)] = a^r b^s \phi^{(r+s)}(ax+by)$

$D^r \phi = \frac{\partial^r \phi}{\partial x^r}$, $D'^s \phi = \frac{\partial^s \phi}{\partial y^s}$, $D^r D'^s \phi = \frac{\partial^{r+s} \phi}{\partial x^r \partial y^s}$

Working rule :

→ To find P.I of an eqn. $F(D,D')z = \phi(ax+by)$ where $F(D,D')$ is a homogeneous function of D, D' of degree proceed as follows :

(1.2)

(i) When $F(a,b) \neq 0$,

we have $P.I = \frac{1}{F(D,D')} \phi(ax+by)$

$= \frac{1}{F(a,b)} \int\int\int \cdots \int \phi(v)\, dv\, dv \cdots dv$

i.e. $P.I = \frac{1}{F(D,D')} \phi(ax+by)$

$= \frac{1}{F(a,b)} \times n^{th}$ integral of $\phi(v)$ w.r.t v

where $v = ax+by$.

(ii) When $F(a,b) = 0$,

Now $F(a,b) = 0$ iff $(bD - aD')$ is a factor of $F(D,D')$.

we have $P.I = \frac{1}{F(D,D')} \phi(ax+by)$

$= \frac{1}{(bD-aD')^n} \phi(ax+by)$

$= \frac{x^n}{b^n n!} \phi(ax+by)$

To find P.I of an eqn $F(D,D')z = x^m y^n$ (or a rational integral algebraic function of x & y).

We have $P.I = \frac{1}{F(D,D')} V$ where $V = x^m y^n$

It evaluated by expanding the symbolic function $\frac{1}{F(D,D')}$ in an infinite series of ascending powers of D' or D

Note:-

If $n < m$, $\frac{1}{F(D,D')}$ should be expanded in powers of $\frac{D'}{D}$, whereas if $m < n$, $\frac{1}{F(D,D')}$ should be expanded in powers of $\frac{D}{D'}$.

problems:

1 → solve $4r - 4s + t = 16 \log(x+2y)$.

sol Given that $4r - 4s + t = 16\log(x+2y)$.

$$\Rightarrow 4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$$

$$\Rightarrow [4D^2 - 4DD' + D'^2]z = 16\log(x+2y) \quad \text{①}$$

A.E of ① is

$$4m^2 - 4m + 1 = 0.$$

$$\Rightarrow (2m-1)^2 = 0$$

$$\Rightarrow m = \tfrac{1}{2}, \tfrac{1}{2}$$

$$\therefore C.F = \phi_1(y + \tfrac{x}{2}) + x\phi_2(y + \tfrac{x}{2})$$

$$= \phi_1\left[\tfrac{1}{2}(2y+x)\right] + x\phi_2\left[\tfrac{1}{2}(2y+x)\right]$$

$$C.F = \psi_1(2y+x) + x\psi_2(2y+x)$$

where ψ_1, ψ_2 are arbitrary f'ns

G.S of ① is $z = C.F + P.I$

Now $P.I = \frac{1}{4D^2 - 4DD' + D'^2} 16\log(x+2y)$

$$= 16\left[\frac{1}{(2D - D')^2}\log(x+2y)\right]$$

$$= 16\left[\frac{x^2}{2^2 \cdot 2!}\log(x+2y)\right] \quad (\because F = 0)$$

$$= 2x^2\log(x+2y).$$

→ solve $(D^2+3DD'+2D'^2)z = x+y$ —— ①

Sol

$$C.F = \phi_1(y-x)+\phi_2(y-2x)$$

where ϕ_1 & ϕ_2 are arbitrary.

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{1^2+3(1)(1)+2(1)^2}\iint f(u)\,du\,du$$

where $u = x+y$

$$= \frac{1}{6}\int \frac{u^2}{2}du$$

$$= \frac{1}{6}\,\frac{u^3}{6}$$

$$= \frac{1}{36}(x+y)^3$$

∴ G.S of ① is $z = C.F + P.I$

$$\Rightarrow z = \phi_1(y-x)+\phi_2(y-2x)+\frac{1}{36}(x+y)^3.$$

→ solve $r-2s+t = \sin(2x+3y)$

→ solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny$.

Sol Given that

$$(D^2+D'^2)z = \cos mx\cos ny \quad —— ①$$

A.E is $m^2+1^2=0$

$$\Rightarrow m = \pm i$$

$$\therefore C.F = \phi_1(y+ix)+\phi_2(y-ix)$$

$$\therefore P.I = \frac{1}{D^2+D'^2}\cos mx\cos ny$$

$$= \frac{1}{D^2+D'^2}\,\frac{1}{2}\left[\cos(mx+ny)+\cos(mx-ny)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+D'^2}\cos(mx+ny)+\frac{1}{D^2+D'^2}\cos(mx-ny)\right] —— ②$$

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Now $\frac{1}{D^2+D'^2}\cos(mx+ny) = \frac{1}{m^2+n^2}\iint \cos u\,du\,du$

$= \frac{1}{m^2+n^2}\int \sin u\,du$

$= \frac{-1}{m^2+n^2}\cos u$

$= \frac{-1}{m^2+n^2}\cos(mx+ny)$

$\frac{1}{D^2+D'^2}\cos(mx-ny) = \frac{-1}{m^2+n^2}\cos(mx-ny)$

$\therefore$ (2) $= P.I. = \frac{1}{D^2+D'^2}\cos mx\cos ny$

$= +\frac{1}{2}\left[\frac{-1}{m^2+n^2}\left(\cos(mx+ny) + \cos(mx-ny)\right)\right]$

$= \frac{-1}{m^2+n^2}\cos mx\cdot\cos ny.$

$\therefore$ G.S. of (1) is $Z = C.F. + P.I.$

$= \phi_1(y+ix) + \phi_2(y-ix) + \frac{1}{m^2+n^2}\cos mx\cdot\cos ny$

1991 → solve $r+5s+6t = (y-2x)^{-1}$

As usual find C.F.

Sol: $P.I. = \frac{1}{D^2+5DD'+6D'^2}(y-2x)^{-1}$

$= \frac{1}{(D+2D')(D+3D')}($

$= \frac{1}{D+2D'}\left\{\frac{1}{D+3D'}(y\right.$

$$= \frac{1}{D+2D'}\left[\frac{1}{-2+3(1)}\int v^{-1}\,dv\right] \quad (16)$$

where $v = y-2x$

$$= \frac{1}{D+2D'}[\log v]$$

$$= \frac{1}{D+2D'}\log(y-2x)$$

$$= \frac{x}{(1)\,1!}\log(y-2x) \qquad (\because f(a,b)=0)$$

$$= x\log(y-2x)$$

$\therefore$ G.S of (1) is $z = C.F + P.I$

→ solve $(D^2-2DD'+D'^2)z = e^{x+2y}$.

→ solve $\log s = x+y$

i.e. $s = e^{x+y}$

1994 → solve $(D^2+3DD'+2D'^2)z = x+y$ by expanding the particular integral (P.I) in ascending powers of D as well as in ascending powers of D'.

Sol) Given that $(D^2+3DD'+2D'^2)z = x+y$ — (1)

A.E of (1) is $m^2+3m+2=0$

$\Rightarrow m = -2, -1$

$\therefore$ $C.F = \phi_1(y-2x) + \phi_2(y-x)$

(17)

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{2D'^2\left[1+\left(\frac{D^2}{2D'^2}+\frac{3}{2}\cdot\frac{D}{D'}\right)\right]}(x+y)$$

$$= \frac{1}{2D'^2}\left[1+\left(\frac{D^2}{2D'^2}+\frac{3}{2}\frac{D}{D'}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{2D'^2}\left[1-\left(\frac{D^2}{2D'^2}+\frac{3}{2}\frac{D}{D'}\right)+\cdots\right](x+y)$$

$$= \frac{1}{2D'^2}\left(x+y-\frac{3}{2}y\right) = \frac{1}{2D'^2}\left(x-\frac{y}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D'}\left[xy-\frac{y^2}{4}\right]$$

$$= \frac{1}{2}\left[\frac{xy^2}{2}-\frac{y^3}{12}\right] = \frac{xy^2}{4}-\frac{y^3}{24}$$

$\therefore$ G.S. of (1) is $z = C.F + P.I$

$$\Rightarrow z = \phi_1(y-x)+\phi_2(y-2x)+\frac{xy^2}{4}-\frac{y^3}{24}$$

Again, by expanding in ascending powers of D', we have

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y) = \frac{1}{D^2\left[1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right]}(x+y)$$

$$= \frac{yx^2}{2}-\frac{x^3}{3}$$

$\therefore$ G.S. of (1) is $z = C.F + P.I$

$$\Rightarrow z = \phi_1(y-2x)+\phi_2(y-x)+\frac{yx^2}{2}-\cdots$$

→ solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$. (18)

Sol: $C.F = \phi_1(y+3x) + x\phi_2(y+3x)$ — ①

$P.I = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2 + 36xy)$

$= \frac{1}{D^2}\left[1 - \left(\frac{6D'}{D} - 9\left(\frac{D'}{D}\right)^2\right)\right]^{-1}(12x^2 + 36xy)$

$= \frac{1}{D^2}\left[1 + \left(\frac{6D'}{D} - 9\left(\frac{D'}{D}\right)^2\right) + \left(\frac{6D'}{D} - 9\left(\frac{D'}{D}\right)^2\right)^2 + \cdots\right](12x^2 + 36xy)$

$= \frac{1}{D^2}\left[(12x^2 + 36xy) + \frac{6}{D}(36x)\right]$

$= \frac{1}{D^2}\left[12x^2 + 36xy + 6(36)\frac{x^2}{2}\right]$

$= \frac{1}{D}\left[4x^3 + 18x^2y + 36x^3\right]$

$= x^4 + 6x^3y + 9x^4$

$= 10x^4 + 6x^3y$

G.S of ① is $z = C.F + P.I$

* A general method finding the P.I:

consider the eqn $(D - mD')z = \phi(x,y)$ — ①

$\Rightarrow p - mq = \phi(x,y)$

∴ Lagrange's A.E's are

$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(x,y)}$ — ②

Taking first two fractions of (1), we get

$dy + m\,dx = 0 \Rightarrow \boxed{y + mx = a}$ (a constant).

Again taking first & last fractions of (1), we get

$$dz = \phi(x, y)\,dx$$

$$\Rightarrow dz = \phi(x, a - mx)\,dx \quad (\because y + mx = a)$$

Integrating we get

$$\boxed{z = \int \phi(x, a - mx)\,dx}$$

$$\text{P.I.} = \frac{1}{D - mD'}\phi(x, y) = z$$

$$= \int \phi(x, a - mx)\,dx \quad \text{--- (3)}$$

where after integration the constant a is to be replaced by $y + mx$ (since the P.I. does not contain any arbitrary constant)

→ Now if the given eqn is $F(D, D')z = \phi(x, y)$

where $F(D, D') = (D - m_1D')(D - m_2D') \cdots (D - m_nD')$

then $\text{P.I.} = \frac{1}{F(D, D')}\phi(x, y)$

$$\Rightarrow \text{P.I.} = \frac{1}{(D - m_1D')(D - m_2D') \cdots (D - m_nD')}\phi(x, y)$$

which can be evaluated by the repeated application of the above method (i.e. (3))

Problems

→ Solve $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \sin x$

The given eqn is $(D + D')z = \sin x$

A.E. is $m + 1 = 0$
$\Rightarrow m = -1$

$\therefore \boxed{\text{C.F.} = \phi_1(y - x)}$

Now P.I $= \frac{1}{D+D'} \sin x$

$= \frac{1}{D+D'} \sin(x+0y)$ | $y - x = c$

$= \int \sin[x + 0(c+x)]\, dx$

$= \int \sin x\, dx$

$= -\cos x.$

$\therefore$ G.S of (1) is $z = C.F + P.I$

1992 → $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \sin x$

Given eqn is $(D^2 + DD' - 6D'^2)z = y \sin x$ —(1)

A.E is $m^2 + m - 6 = 0 \Rightarrow m = 2, -3.$

$\therefore$ $C.F = \phi_1(y+2x) + \phi_2(y-3x)$

$P.I = \frac{1}{D^2+DD'-6D'^2}(y \sin x) = \frac{1}{(D-2D')(D+3D')}(y \sin x)$

$= \frac{1}{D-2D'}\left[\frac{1}{D-(-3)D'}(y \sin x)\right]$

$= \frac{1}{D-2D'}\left[\int (c+3x) \sin x\, dx\right]$ where $y-3x = c$

$= \frac{1}{D-2D'}\left[-y \cos x + 3 \sin x\right]$

$= \int \left[-(c-2x)\cos x + 3 \sin x\right] dx$ where $y+2x = c$

$= -y \sin x - \cos x.$

$\therefore$ G.S of (1) is $z = C.F + P.I$

→ solve $r - t = \tan^3 x \tan y - \tan x \tan^3 y.$

2004 → solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x$

2000 → Solve $(D^2 - DD' - 2D'^2)z = e^{2x+3y} + e^{x+4y}$

2008, 12M → Solve $(D^2 + DD' - 6D'^2)z = y \cos x$

Non-homogenious linear partial Differential eqns with constant coefficients:

A linear partial differential eqn which is not homogeneous, is called a non-homogenious linear eqn.

consider the diff. eqn $F(D,D')z = f(x,y)$

when $F(D,D')$ is a homogeneous function of D, D' it can always be resolved into linear factors. But the result is not always true when $F(D,D')$ is non-homogeneous.

Now we classify linear differential operators $F(D,D')$ into two types.

These are: (i) $F(D,D')$ is reducible if it can be written as the product of linear factors of the form $D+aD'+b$, with a, b constants.

(ii) $F(D,D')$ is irreducible if it cannot be so written.

→ (i) **C.F of non-homogeneous linear eqn when $F(D,D')$ can be resolved into linear factors:**

The C.F of non-homo. linear eqn (1) is the g.s of the eqn $F(D,D')z=0$ — (2).

Let us consider a simple non-homo. eqn $(D-mD'-k)z=0$

$\Rightarrow p - mq = kz$ — (3)

Lagrange's A.E's are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{kz}$$

Taking the first two fractions of ①, we get

$dy + m\,dx = 0 \Rightarrow \boxed{y + mx = a}$ (const).

Again taking the first & the last fraction of ①, we get

$\frac{dz}{z} = k\,dx$

$\Rightarrow \log z = kx + \log b$

$\Rightarrow \boxed{z = b e^{kx}}$

$\Rightarrow z = e^{kx}\phi(a)$ $(\because b = \phi(a))$

$\Rightarrow \boxed{z = e^{kx}\phi(y + mx)}$

which is the soln of ①.

→ If $F(D, D')$ can be factorised into non-repeated linear factors $(D - m_1D' - k_1), (D - m_2D' - k_2), \ldots, (D - m_nD' - k_n)$ then the eqn ② becomes $[(D - m_1D' - k_1)(D - m_2D' - k_2) \ldots (D - m_nD' - k_n)]z = 0$ — ⑥

∴ G.S of ② is

$z = e^{k_1x}\phi_1(y + m_1x) + e^{k_2x}\phi_2(y + m_2x) + \ldots + e^{k_nx}\phi_n(y + m_nx)$.

NOTE:– If the eqn is $(D + p_1D' + r)z = 0$ then

$z = e^{-rx/2}\,\phi\,(p_1x - 2y)$.

→ $F(D, D')$ has repeated factors

(i) If $(D - mD' - k)$ occurs twice then the g.s of ② is

$z = e^{kx}[\phi_1(y + mx) + x\,\phi_2(y + mx)]$.

(ii) If $(D - mD' - k)$ occurs r times then the g.s of ② is

$z = e^{kx}[\phi_1(y + mx) + x\,\phi_2(y + mx) + \ldots + x^{r-1}\phi_r(y + mx)]$.

(ii) When linear factors of $F(D,D')$ are not possible:

In case $F(D,D')$ is irreducible, i.e. it can not be resolved into linear factor in D & D', the above methods of finding the complementary function fails. In such cases a trial method is used to find solns.

Ex:- Solve $(2D^4 - 3D^2D' + D'^2)z = 0$

$\Rightarrow (2D^2 - D')(D^2 - D')z = 0$ ——①

Let $z = Ae^{hx+ky}$ ——② be the soln corresponding to $(D^2 - D')z = 0$ ——③

$\Rightarrow D^2[Ae^{hx+ky}] - D'[Ae^{hx+ky}] = 0$

$\Rightarrow Ah^2e^{hx+ky} - AKe^{hx+ky} = 0$

$\Rightarrow A(h^2 - K)e^{hx+ky} = 0$

$\Rightarrow h^2 - K = 0$ $\quad (Ae^{hx+ky} \neq 0)$

$\Rightarrow \boxed{K = h^2}$

putting $K = h^2$ in ②, we get

$z = Ae^{hx+h^2y}$

Since all values of h satisfy it, the more general soln of

$z = \sum Ae^{hx+\ldots}$

Sly the g.s of $(2D^2 - D')z = 0$ is

$$z = \sum B e^{h^2 x + 2h^2 y} \quad \text{(5)}$$

∴ The most g.s of the given eqn (1)

is $z = \sum A e^{hx+hy} + \sum B e^{h^2 x + 2h^2 y}$

→ solve $(D - D'^2)z = 0$

→ solve $(D^2 - D'^2 + D - D')z = 0$

Sol Given that $(D^2 - D'^2 + D - D')z = 0$ — (1)

$\Rightarrow (D - D')(D + D' + 1)z = 0$

∴ There are distinct factors.

∴ G.S of (1) is

$z = e^{0x}\phi_1(y+x) + e^{-x}\phi_2(y-x)$

→ solve $DD'(D - 2D' - 3)z = 0$ — (1)

Sol There are three distinct factors.

∴ The g.s of (1) is

$z = \phi_1(y) + \phi_2(x) + e^{3x}\phi_3(y+2x)$

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Particular Integral:

The complete soln of $F(D,D')z = f(x,y)$

is $z = C.F + P.I$,

where $P.I = \frac{1}{F(D,D')} f(x,y)$. — (1)

The methods of obtaining particular integrals of non-homo. PDE's are very similar to those of ordinary linear eqns with constant coefficients.

→ We now give some cases of finding the particular integrals.

Case (i) If $f(x,y) = e^{ax+by}$ and $F(a,b) \neq 0$.

then $P.I = \frac{1}{F(D,D')} e^{ax+by}$

$= \frac{1}{F(a,b)} e^{ax+by}$.

Case (ii) If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

then $P.I = \frac{1}{F(D,D')} \sin(ax+by)$ is evaluated by putting $D^2 = -a^2$, $DD' = -ab$ & $D'^2 = -b^2$,

provided the denominator is not 0.

Case (iii) If $f(x,y) = x^m y^n$ then

$P.I = \frac{1}{F(D,D')} (x^m y^n) = [F(\ldots$

which can be evaluated af. $[F(D,D')]^{-1}$ in ascending powers of

case (iv) If $f(x,y) = e^{ax+by} \cdot V$ where V is fn of x & y.

then $P.I = \frac{1}{F(D,D')} e^{ax+by} \cdot V$

$$= e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)} V.$$

1991 → Solve $s + p - q = z + xy$

Given that $\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = z + xy$ —— (1)

$\Rightarrow (DD' + D - D' - 1) z = xy$

$\Rightarrow (D-1)(D'+1) z = xy$ —— (2)

∴ There are two distinct linear factors

ie $(D-1)(D'+1) z = 0$.

∴ C.F of (1) is

$e^x \phi_1(y) + e^{-y} \phi_2(x)$, where ϕ_1 & ϕ_2 are arbitrary

Now $P.I = \frac{1}{(D-1)(D'+1)} (xy)$

$$= -(1-D)^{-1} (1+D')^{-1} (xy)$$

$$= -[(1 + D + D^2 + D^3 \dots) (1 - D' + D'^2 - D'^3 \dots)](xy)$$

$$= -[1 - D' + D - DD' + \dots](xy)$$

$$= -[xy - x(1) + y(1) - (1)]$$

$$= -[xy - x + y - 1]$$

∴ The reqd g.s is $z = C.F + P.I$.

ie $z = e^x \phi_1(y) + e^{-y} \phi_2(x) - (xy + y - x - 1)$

1997 → Solve $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$

1998 → Solve $r - s + 2q - z = x^2y^2$

2005 → Solve $(D - 3D' + 2)^2 z = 2e^{2x}\sin(y+3x)$

2007 → Solve $(D^2 - DD' - 3D + 3D')z = xy + e^{x+2y}$

→ Solve $r - s + p = 1$.

→ Solve $(D - 3D' - 2)^2 z = 2e^{2x}\tan(y+3x)$ — (1)

$$C.F = e^{2x}\left[\phi_1(y+3x) + x\phi_2(y+3x)\right]$$

$$P.I = \frac{1}{(D-3D'-2)^2} 2e^{2x+0y}\tan(y+3x)$$

$$= 2e^{2x}\cdot\frac{1}{[D+2-3(D'+0)-2]^2}\tan(y+3x)$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

$$= 2e^{2x}\cdot\frac{x^2}{1^2\cdot 2!}\tan(y+3x) \quad \left(\frac{1}{(bD-aD')^2}\phi(ax+by) = \frac{x^2}{b^2 2!}\phi(ax+by)\right)$$

$$= x^2e^{2x}\tan(y+3x)$$

∴ The reqd G.S of (1) is $z = C.F + P.I$

Eqns reducible to linear form with constant coefficients :

A PDE having variable co-efficients can sometimes be reduced to an eqn with constant coefficients by suitable substitutions.

Reduce an eqn of the form

$$A_0 x^n \frac{\partial^n z}{\partial x^n} + A_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 x^{n-2} y^2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots$$

$$\cdots\cdots = f(x,y) \quad \text{--- (1)}$$

into a linear eqn with constant coefficients.

Note: In the eqn (1), the term $\frac{\partial^n z}{\partial x^r \partial y^{n-r}}$ is multiplied by the variable expression $x^r y^{n-r}$.

To transform the eqn (1),

putting $x = e^X$, $y = e^Y$

$\Rightarrow$ $\boxed{X = \log x}$; $\boxed{Y = \log y}$

Now $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{1}{x}\cdot\frac{\partial z}{\partial X}$

$\therefore$ $\boxed{x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}}$

$\therefore$ $x\frac{\partial}{\partial x} = \frac{\partial}{\partial X} = D$ (say) --- (2)

Now $x\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right) = x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x}$

$\Rightarrow$ $x^2\frac{\partial^2 z}{\partial x^2} = \left(x\frac{\partial}{\partial x} - 1\right)x\frac{\partial z}{\partial x}$

$= (D-1)Dz$

$= D(D-1)z$ --- (3)

In general $x^n \frac{\partial^n z}{\partial x^n} = D(D-1)(D-2)\cdots(D-n+1)z$ — (A)

Now $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{1}{y} \frac{\partial z}{\partial Y}$

$$\Rightarrow \boxed{y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}}$$

$$\therefore y \frac{\partial}{\partial y} = \frac{\partial}{\partial Y} = D' \text{ (say)}$$

Sly $y^2 \frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$

In general, $y^n \frac{\partial^n z}{\partial y^n} = D'(D'-1)\cdots(D'-n+1)z$

Also $xy \frac{\partial^2 z}{\partial x \partial y} = DD'z$

and $x^m y^n \frac{\partial^{m+n} z}{\partial x^m \partial y^n} = D(D-1)\cdots(D-m+1)D'(D'-1)\cdots(D'-n+1)z$

These substitutions reduce the eq (1) to an eqn having constant coefficients and now it can be easily solved by the known methods of homo & non-homo. linear eqns with constant coefficients.

Problems 1987: Solve $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$ — (1)

Sol: putting $x = e^X$, $y = e^Y$ and denoting the operators $\frac{\partial}{\partial X}$ & $\frac{\partial}{\partial Y}$

$\therefore (1) \equiv [D(D-1) + 2DD' +$

$\Rightarrow [D^2 - D + 2DD' + D'^2 -$

$\Rightarrow (D+D')(D+D' -$

$\therefore$ G.S. of (1) is $z = \phi_1(Y$

$\therefore z = \phi_1(\log y - \log x) + x\,\phi_2(\log y - \log x)$

$= \phi_1(\log(\frac{y}{x})) + x\,\phi_2(\log(\frac{y}{x}))$ $(\because x = e^X \;\&\; y = e^Y)$

$= f_1(\frac{y}{x}) + x f_2(\frac{y}{x})$ which is the req. g.s. of ①

1997 → Solve $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = xy$

putting $x = e^X$; $y = e^Y$ ——①

and denoting the operators $\frac{\partial}{\partial X}$ & $\frac{\partial}{\partial Y}$ by D & D'

$\therefore ① \equiv [D(D-1) - D'(D'-1)]z = e^{X+Y}$

$\Rightarrow [D^2 - D'^2 - D + D']z = e^{X+Y}$

$\Rightarrow (D-D')(D+D'-1)z = e^{X+Y}$ ——②

$\therefore C.F = \phi_1(Y+X) + e^X \phi_2(Y-X)$

$= \phi_1(\log y + \log x) + x\,\phi_2(\log y - \log x)$ $(\because x = e^X \;\&\; y = e^Y)$

$= \phi_1(\log(xy)) + x\,\phi_2(\log(\frac{y}{x}))$

$= f_1(xy) + x f_2(\frac{y}{x})$

Now $P.I = \frac{1}{(D-D')(D+D'-1)} e^{X+Y}$

$= \frac{1}{(D-D')(1+1-1)} e^{X+Y}$

$= \frac{1}{D-D'} e^{X+Y}$

$= \frac{X}{1!} e^{X+Y}$

$= (\log x)\, xy$

$\therefore$ G.S of ① is $z = C.F + P.I$

1993 / 1997 → Solve $x^2 r - y^2 t + px - qy = \log x$.

2001 → Solve the eqn $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \log x$ by reducing it to the eqn with const. coefficients.

Cauchy's Problem for Second Order Partial Differential equation. Characteristic equation and Characteristic Curves (or simply characteristics) of the second-order Partial Differential Equations.

Cauchy's Problem. Consider the second order Partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{---- (1)}$$

in which R, S and T are functions of x and y only. The Cauchy's Problem consists of the Problem of determining the solution of (1) such that on a given space Curve C it takes on prescribed values of z and $\partial z/\partial n$, where n is the distance measured along the normal to the Curve.

As an example of Cauchy's Problem for the Second order Partial differential equation, consider the following Problem:-

To determine solution of $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ with the following data prescribed on the x-axis: $z(x, 0) = f(x)$, $z_y(x, 0) = g(x)$. Observe that y-axis is the normal to the given Curve (x-axis here)

Characteristic equations and Characteristic Curves.

Corresponding to (1), Consider the λ-quadratic.

$$R\lambda^2 + S\lambda + T = 0 \quad \text{---- (2)}$$

when $S^2 - 4RT \geq 0$, (2) has real roots. then

differential equation $(dy/dx) + \lambda(x,y) = 0$ ——— (3)

are called the characteristic equations.

The solutions of (3) are known as characteristic curves or simply the characteristics of the second order partial differential equation (1).

Now, consider the following three cases.

Case(i): If $S^2 - 4RT > 0$ (i.e if (1) is hyperbolic), then (2) has two distinct real roots λ_1, λ_2 say so that we have two characteristic equations

$$\left(\frac{dy}{dx}\right) + \lambda_1(x,y) = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right) + \lambda_2(x,y) = 0$$

solving these we get two distinct families of characteristics.

Case(ii). If $S^2 - 4RT = 0$ (i.e (1) is parabolic), then (2) has two equal real roots λ, λ so that we get only one characteristic equation (3). solving it, we get only one family of characteristics.

Case(iii): If $S^2 - 4RT < 0$ (i.e. (1) is elliptic), then (2) has complex roots. Hence there are no real characteristics. Thus we get two families of complex characteristics when (1) is elliptic.

2009 / 124 → Find the characteristics of $y^2r - x^2t = 0$

sol'n: Given $y^2r - x^2t = 0$ ——— (1)

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

here $R = y^2$, $S = 0$ and $T = -x^2$.

Then $S^2 - 4RT = 0 - 4 \cdot y^2(-x^2)$

$$= 4x^2y^2 > 0$$

and hence (1) is hyperbolic everywhere except on the Coordinate axes $x=0$ and $y=0$.

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ (or)

$$y^2\lambda^2 - x^2 = 0 \quad \text{--- (2)}$$

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots)

corresponding characteristic equations are.

$$(dy/dx) + (x/y) = 0 \quad \text{and} \quad (dy/dx) - (x/y) = 0$$

$$x\,dx + y\,dy = 0 \quad \text{and} \quad x\,dx - y\,dy = 0$$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$.

which are the required families of characteristics.

Here these are families of Circles and hyperbolas respectively.

→ Find the characteristics of $x^2r + 2xys + y^2t = 0$.

Sol'n: Given $x^2r + 2xys + y^2t = 0$ —— (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

here $R = x^2$, $S = 2xy$ and $T = y^2$.

Then, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$

and hence (1) is parabolic everywhere

the λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ or x^2

solving (2), $(x\lambda + y)^2 = 0$ so that $\lambda = -y/x, -y/x$ (equal roots)

The characteristic equation is $(dy/dx) - (y/x) = 0$

(or) $\left(\frac{1}{y}\right) dy - \left(\frac{1}{x}\right) dx = 0$ giving $y/x = c_1$ (or) $y = c_1 x$

which is the required family of characteristics.

Here it represents a family of straight lines passing through the origin.

H.W

→ Find the characteristics of $4r + 5s + t + p + q - 2 = 0$

[Ans: $y - x = c_1$, and $y - (x/4) = c_2$]

→ Find the characteristics of $(\sin^2 x) r + (2\cos x) s - t = 0$.

[Ans: $y + \operatorname{cosec} x - \cot x = c_1$, $y + \operatorname{cosec} x + \cot x = c_2$]

Applications of Differential Equations

1.

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions.

The differential equation together with these boundary conditions, constitute a boundary value problem.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions.

Most of the boundary value problems involving linear partial differential equations can be solved by the method of separation of variables.

Method of seperation of variables (or) product method

- It involves a solution which breaks up into a product of functions each of which contains only one of the variables.

The following example explains this method.

→ Solve (by the method of seperation of variables):

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Sol: Assume the trial solution $z = X(x)\,Y(y)$ —— (1)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0$$

where $X' = \frac{dX}{dx}$, $Y' = \frac{dY}{dy}$ etc.

Seperating the variables, we get

$$(X'' - 2X')Y + XY' = 0$$

$$\Rightarrow \frac{X'' - 2X'}{X} = \frac{-Y'}{Y} \quad \text{—— (i)}$$

Since x and y are independent variables, therefore (i) can only true if each side is equal to the same constant, a (say)

$$\therefore \frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = a$$

$$\Rightarrow \frac{X'' - 2X'}{X} = a \quad \text{and} \quad \frac{-Y'}{Y} = a$$

i.e, $X'' - 2X' - Xa = 0$ —— (ii)   i.e $Y' + aY = 0$ —— (iii)

To solve the equation (ii),

the auxiliary equation is

$$m^2 - 2m - a = 0$$

$$\Rightarrow m = 1 \pm \sqrt{1+a}$$

$\therefore$ the solution of (ii) is

$$X = C_1 e^{(1+\sqrt{1+a})x} + C_2 e^{(1-\sqrt{1+a})x}$$

and the solution of (iii) is $Y = C_3 e^{-ay}$

Substituting these values of X and Y in (i), we get

$$z = \left\{ C_1 e^{(1+\sqrt{1+a})x} + C_2 e^{(1-\sqrt{1+a})x} \right\} C_3 e^{-ay}$$

i.e, $z = k_1 e^{(1+\sqrt{1+a})x} + k_2 e^{(1-\sqrt{1+a})x}$

where $k_1 = C_1 C_3$ and $k_2 = C_2 C_3$

which is the required complete sol.

→ Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where ...

Sol: Assume the solution $u(x,t) =$

Substituting in the given equation, we have

$$X'T = 2XT' + XT$$

$$\Rightarrow (X' - X)T = 2XT'$$

$$\Rightarrow \frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$$\therefore \frac{X' - X}{2X} = k \quad \text{and} \quad \frac{T'}{T} = k \quad \text{--- (iii)}$$

$$\Rightarrow X' - X - 2kX = 0$$

$$\Rightarrow X' = (1+2k)X$$

$$\Rightarrow \frac{X'}{X} = (1+2k) \quad \text{--- (ii)}$$

Solving (ii) $\log X = (1+2k)x + \log c$

$$\Rightarrow \boxed{X = ce^{(1+2k)x}}$$

From (iii) $\log T = kt + \log c'$

$$\Rightarrow \boxed{T = c'e^{kt}}$$

Thus from (i), we have

$$u(x,t) = XT = cc'e^{(1+2k)x} \cdot e^{kt} \quad \text{--- (iv)}$$

Given that $u(x,0) = 6e^{-3x}$

$\therefore$ from (iv) $u(x,0) = 6e^{-3x} = cc'e^{(1+2k)x}$

$$\Rightarrow \boxed{cc' = 6} \text{ when } 1+2k = -3$$

$$\Rightarrow \boxed{k = -2}$$

Substituting these values in (iv), we get

$$u(x,t) = 6e^{-3x}e^{-2t} \quad \text{i.e.} \quad \boxed{u = 6e^{-(3x+2t)}}$$

which is the required solution.

Note: Suppose the given partial differential equation involves n independent variables

$x_1, x_2, \ldots x_n$, and one dependent variable 'u'. Then assume that the equation possesses product solution of the form

$$u(x_1, x_2, \ldots x_n) = X_1(x_1) \cdot X_2(x_2) \ldots X_n(x_n) \quad \text{——— (1)}$$

where X_i is a function of x_i only $(i = 1, 2, \ldots n)$.

On substitution of (1) into the given equation, we shall obtain 'n' ordinary differential equations one in each of the unknown functions X_i $(i = 1, 2, \ldots n)$.

Solve the following equations by the method of separation of variables:

(1) $py^3 + qx^2 = 0$

(2) $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$

(3) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$

(4) Find a solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} + 2u$ in the form $u = f(x) g(y)$. Solve the equation subject to the conditions $u = 0$ and $\frac{\partial u}{\partial x} = 1 + e^{-3y}$, when $x = 0$ for all values of y.

The following are the well-known partial differential equations:

(i) wave equation : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (or) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

(ii) One dimensional heat flow equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(iii) Two dimensional heat flow equation which in steady state becomes the two dimensional Laplace's equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(iv) vibrating membrane : Two dimensional wave equation i.e, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$.

(v) Laplace's equation in three dimensions

i.e, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. etc.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solutions gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

Some basic definitions

Rest: A body is said to be at rest if it doesnot change its position with time with respect to its surroundings.

Motion: A body is said to be in motion if it changes its position with time with respect to its surroundings.

- Terms rest and motion are relative to each other.

For examples when a train is running, two passengers sitting in the train beside each other are at rest with respect to each other but they are in motion with respect to the person standing outside the train.

Displacement: the shortest distance between the starting point to the ending point is called 'displacement'. It is a vector.

(or)

Displacement is a vector quantity representing a change of position.

Deflection: A sudden change in the direction that something is moving in.

Distance: Total length of the path covered by

velocity: Displacement of a body per is said to be its velocity.

If 'S' is displacement that takes place in time 't' then velocity of the body is given as

$$\text{velocity} = \frac{\text{displacement}}{\text{time taken}}$$

Acceleration:- If velocity of a body changes with time (either due to change in magnitude or direction or both) it is said to have acceleration.

i.e, the rate of change of velocity is called acceleration.

Equilibrium: A system of forces acting on a particle is said to be in equilibrium if it is either at rest or moves with uniform motion in a straight line.

Mass: Mass of a body is the quantity of matter it contains.

Force: Force is an external agency which changes or tends to change the state of rest or of uniform motion in a straight line.

The effect of force acting on a rigid body depends not only on magnitude but also on its direction and point of application.

Weight of body:

The force with which a body is attracted towards the centre of the earth due to the gravitational attraction is called the weight of the body on the earth.

$W = mg$.

where m is mass of body and g is acceleration due to gravity for earth.

Mass of the body remains constant at any place but weight of the body varies with changes in 'g'.

Gravitational force:

Gravitational force is a long range force and is responsible for the attraction between particles of different masses in universe.

Nature of gravity:

The gravitational force between two bodies is always attractive. It depends upon the masses of the bodies and the distance between them. The greater the masses the is the force, the greater the dist seperation the lesser is the force. Independent of the nature of the between the bodies.

→ Tension is a pulling force which is exerted on a body by means of a string or rod.

→ Thrust is a pushing force which is exerted on a body by means of a rod and not by means of string, because a string is flexible.

Newton's Laws of Motion

Newton's First Law:

Every body continues to be in its state of rest or of uniform motion along a straight line unless it is acted on by an external force to change its state.

Newton's Second Law:

The rate of change of momentum of a body is directly proportional to the external force applied and it takes place in the same direction in which the external force is acting.

$$F = ma$$

Momentum: The momentum (P) of a body is defined as the product of its mass (m) and velocity (v).

$$P = mv$$

Newton's third Law:

To every action there is always an equal and opposite reaction.

$$F_{(action)} = F_{(reaction)}$$

→ Action and reaction are equal in magnitude and opposite in direction.

→ They always occur in pairs.

principle of superposition of waves:

— principle of superposition of waves states that when two or more waves are simultaneously impressed on the particles of the medium, the resultant displacement of any particle is equal to the algebraic sum of displacements of all the waves.

If y_1, y_2, y_3 ... etc are the displacements due to the overlapping waves, the resultant displacement of any particle is given by

$$y = y_1 + y_2 + y_3 + \cdots$$

Thus the resultant wave form can be obtained by the principle of superposition of waves.

— The general linear homogeneous partial differential equation of the second order is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \quad \text{--- (1)}$$

Suppose that (i) $u_1, u_2, u_3, \ldots u_n \ldots$ is an infinite set of solutions of (1) in a region R in xy-plane

(ii) The infinite series $u_1 + u_2 + \cdots$ converges and is differentiable term by term in R. Then, by principle the function u, defined by -

a solution of (1) in R. Here R denotes the set of all real numbers.

Fourier sine series: If it be required to expand $f(x)$ as a sine series in $0 < x < l$, then its expansion will give the Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx, \quad n = 1, 2, 3, \ldots$$

Also known as half range sine series.

Fourier cosine series: If it be required to expand $f(x)$ as a cosine series in $0 < x < l$, then its expansion will give the Fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{where } a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\, dx, \quad n = 1, 2, 3, \ldots$$

Also known as half-range cosine series.

Double Fourier sine series:
If it be required to expand $f(x, y)$ as a sine series in rectangle $0 \le x \le a$, $0 \le y \le b$, then its expansion will give double Fourier sine series.

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where $A_{mn} = \dfrac{4}{ab} \displaystyle\int_{x=0}^{a} \int_{y=0}^{b} f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy$, $m = 1, 2, 3, \ldots$; $n = 1, 2, 3, \ldots$

Triple Fourier sine series: If it be required to expand $f(x, y, z)$ as a sine series in parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, then its expansion will give triple Fourier sine series.

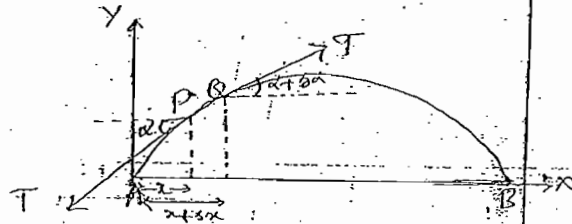
$$f(x, y, z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{lmn} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c},$$

where $A_{lmn} = \dfrac{8}{abc} \displaystyle\int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} f(x, y, z) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}\, dx\, dy\, dz$, $l = 1, 2, 3, \ldots$; $m = 1, 2, 3, \ldots$; $n = 1, 2, 3, \ldots$

Vibrations of stretched elastic string wave Equation

Consider a tightly stretched elastic string of length l and fixed ends A and B subjected to constant tension T as shown in the figure.

The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Y, T, P, Q, α, $\alpha+\delta\alpha$, T, A, x, $x+\delta x$, B, X

Let the string be released from rest and allowed to vibrate.

We shall study the subsequent motion of the string, with no external forces acting on it assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

Taking the end A as the origin, AB as the x-axis and AY perpendicular as the y-axis, so that the motion takes place entirely in the xy-plane. Let the string in the position APB

[illegible] the motion of the element PQ of the [illegible] the points $P(x,y)$ and $Q(x+\delta x, y+\delta y)$, [illegible] tangents makes angles α and $\alpha+\delta\alpha$ [illegible].

[illegible] the element is moving upwards with [acce]leration $\frac{\partial^2 y}{\partial t^2}$.

[S]ince there is no motion in horizontal direction, we have

$$T\cos(\alpha+\delta\alpha) - T\cos\alpha = 0$$

$$\Rightarrow T\cos(\alpha+\delta\alpha) = T\cos\alpha = T \text{ (say) constant}$$

Also the vertical component of the force acting on this element PQ is $= T\sin(\alpha+\delta\alpha) - T\sin\alpha$

$$= T\{\sin(\alpha+\delta\alpha) - \sin\alpha\}$$

$$= T\{\tan(\alpha+\delta\alpha) - \tan\alpha\}, \quad (\because \alpha \text{ is small})$$

$$= T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}$$

If m be the mass per unit length of the string, then mass of the element PQ is $= m\delta x$.

Then by Newton's second law of motion we have

$$m\,\delta x \cdot \frac{\partial^2 y}{\partial t^2} = T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}$$

i.e. $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left\{\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right\}$$

Taking limits as $Q \to P$, i.e. $\delta x \to 0$,

we have $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

i.e. $$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}} \quad \text{where } \boxed{c^2 = \frac{T}{m}}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional wave equation.

Solution of one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Sol: Given $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— (1)

Let solution of (1) be of the form

$y(x,t) = X(x)\,T(t)$ —— (2)

where X is a function of x and T a function of t only.

Then $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$.

Substituting these values in (1), we get

$$XT'' = c^2 X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \quad —— (3)$$

Clearly the left side of (3) is a function of x only and the right side is a function of t only.

Since x and t are independent variables (3) can hold good if each side is equal to a constant k (say).

Then (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2T}{dt^2} - kc^2T = 0 \quad —— (4)$$

Now we solve (4) and (5); Three cases arise

(i) when $k=0$. Then $X = a_1 x + a_2$; $T = a_3 t + a_4$

(ii) when k is positive.

Let $k = p^2$ (say)

Then $X = b_1 e^{px} + b_2 e^{-px}$; $T = b_3 e^{cpt} + b_4 e^{-cpt}$

(iii) when k is negative.

Let $k = -p^2$ (say).

Then $X = c_1 \cos px + c_2 \sin px$;

$T = c_3 \cos cpt + c_4 \sin cpt$

Thus the various possible solutions of wave equation (1) are

$$y(x,t) = (a_1 x + a_2)(a_3 t + a_4) \quad \text{--- (6)}$$

$$y(x,t) = (b_1 e^{px} + b_2 e^{-px})(b_3 e^{cpt} + b_4 e^{-cpt}) \quad \text{--- (7)}$$

and $$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt). \quad \text{--- (8)}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence $y(x,t)$ must involve trigonometric terms. Accordingly the solution given by (8) i.e.,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

is the only suitable solution of the wave equation.

Boundary conditions and initial conditions of one dimensional wave equation:

→ The boundary conditions which the solution has to satisfy are

(i) $y(x, t) = 0$ when $x = 0$

(ii) $y(x, t) = 0$ when $x = l$ (where 'l' is the length of the stretched string)

These should satisfy for every value of t.

i.e, As the end points of the string are fixed, for all time :-

$y(0, t) = 0$ and $y(l, t) = 0$

→ If the string is made to vibrate by pulling it in a curve $y = f(x)$ and then releasing it, the initial conditions are

$y(x, t) = f(x)$ when $t = 0$ i.e, $y(x, 0) = 0$ and

$\frac{\partial y}{\partial t}(x, t) = 0$ when $t = 0$ i.e, $\left(\frac{\partial y}{\partial t}\right)_t = 0$

Here the initial velocity of the string is zero i.e, the string starts from the position of rest.

→ If the string is made to vibrate by giving its each point (when in equilibrium position) a specified velocity, the initial co are of the form

$y(x,t) = 0$ when $t=0$ i.e., $y(x,0)=0$

$\frac{\partial y(x,t)}{\partial t} = g(x)$ when $t=0$ i.e., $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$

→ If the string is given both a displacement and velocity initially, then the initial conditions are of the form

$y(x,t) = f(x)$ when $t=0$ i.e. $y(x,0) = f(x)$

$\frac{\partial y}{\partial t}(x,t) = g(x)$ when $t=0$ i.e. $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$

General solution of one-dimensional wave equation satisfying the given boundary and initial conditions.

Ex ① Show that the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ under conditions $y(0,t)=0$, $y(l,t)=0$ $\forall t$.

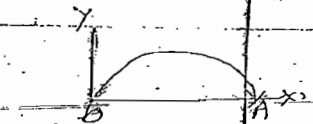
$y(x,0) = f(x)$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$

has solution of the form:

$$y(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l} \right) \sin\frac{n\pi x}{l}$$

where $E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$

$$F_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx$$

Sol^n: Given $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ——— ①

where $u(x,t)$ is the deflection of the string.

Let it be stretched between fixed points $(0,0)$ and $(l,0)$. Then we are to find $y(x,t)$ under the following boundary condition (B.C) and initial conditions (I.C):

B.C. $y(0,t) = 0$, $y(l,t) = 0$ for all t ——— ②

I.C. $y(x,0) = f(x)$ (Initial deflection) ——— ③

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$ (Initial velocity) ——— ④

Suppose that ① has the solution of the form $y(x,t) = X(x)\, T(t)$

Substituting this value of u in ①, we have

$$XT'' = c^2 X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu \text{ (say)}$$

$$\Rightarrow X'' - \mu X = 0 \quad \text{—— (6)}$$

and $T'' - \mu c^2 T = 0$ —— (7)

Using ②, ③ gives

~~$X(0)\,T(t) = 0$ and $X(l)\,T(t) = 0$~~ ⑧

Since $T(t) \equiv 0$ leads to $y \equiv 0$.

So suppose ~~that~~ $T(t) \not\equiv 0$.

Then ⑧ gives

$\boxed{X(0) = 0}$ and $\boxed{X(l) = 0}$ —— ⑨ which are boundary conditions

We now solve ⑥ under B.C. ⑨.

Three cases arise.

Case ①: Let $\mu = 0$. Then solution of ⑥ is

$$X(x) = Ax + B \quad \text{—— (10)}$$

Using B.C. ⑨, ⑩ gives

$X(0) = A(0) + B \Rightarrow \boxed{0 = B}$

and: $X(l) = Al + B \Rightarrow 0 = Al + 0$

$\Rightarrow \boxed{A = 0}$ (l - l is the length of string)

$\Rightarrow \boxed{X(x) = 0}$

This leads to $y \equiv 0$, which doesnot satisfy I.C. ③ and ④.

So we reject $\mu = 0$.

Case(2): Let $\mu = \lambda^2$, $\lambda \neq 0$ (i.e., μ is positive). Then solution of (8) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{——— (11)}$$

Using B.C. (9), (11) gives

$$X(0) = 0 = A + B \quad \text{i.e.,} \quad \boxed{A + B = 0} \quad \text{——— (12)}$$

and $X(l) = 0 = Ae^{\lambda l} + Be^{-\lambda l}$

solving above we get

$$Ae^{\lambda l} - Ae^{-\lambda l} = 0$$

$$\Rightarrow A(e^{\lambda l} - e^{-\lambda l}) = 0 \quad (\because e^{\lambda l} - e^{-\lambda l} \neq 0, \ l, \lambda \neq 0)$$

$$\Rightarrow \boxed{A = 0}$$

$\therefore$ from (12) $\boxed{B = 0}$

$$\Rightarrow \boxed{X(x) = 0}$$

This leads to $y \equiv 0$ which does not satisfy (3) and (4).

So we reject $\mu = \lambda^2$.

Case(3): Let $\mu = -\lambda^2$, $\lambda \neq 0$ (i.e., μ is negative)

Then solution of (8)

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{——— (13)}$$

Using B.C. (9), (13) gives

$$X(0) = 0 = A(1) + B(0) \Rightarrow \boxed{A = 0}$$

and $X(l) = 0 = 0 + B\sin\lambda l$

$$\Rightarrow B\sin\lambda l = 0$$

$$\Rightarrow \sin\lambda l = 0$$

Here we taken B ~~≠~~

Since otherwise $X \equiv 0$

which does not sa

Now $\sin\lambda l = 0$

$\Rightarrow \lambda l = n\pi$, $n = 1, 2, \ldots$

$\Rightarrow \boxed{\lambda = \frac{n\pi}{l}}$, $n = 1, 2, 3, \ldots$

$\therefore$ from (13), we have

$X(x) = B \sin \frac{n\pi x}{l}$, $n = 1, 2, 3, \ldots$

Hence non-zero solutions $X_n(x)$ of (8) are given by

$$\boxed{X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)} \quad \text{--- (14)}$$

from (9)

$T'' - \mu c^2 T = 0$

$\Rightarrow T'' + \lambda^2 c^2 T = 0 \quad (\because \mu = -\lambda^2)$

$\Rightarrow T'' + \frac{n^2\pi^2 c^2}{l^2} T = 0 \quad (\because \lambda = \frac{n\pi}{l})$

whose general solution is

$$T_n(t) = C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right)$$

$\therefore y_n(x, t) = X_n(x) \cdot T_n(t)$

$$= B_n \sin \frac{n\pi x}{l}\left[C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l}\right]$$

$$= \left[E_n \cos \frac{n\pi ct}{l} + F_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l} \quad \text{--- (A)}$$

are solutions of (1) satisfying (2).

Here $E_n = B_n C_n$ and $F_n = B_n D_n$ are new arbitrary constants.

In order to obtain a solution also satisfying (3) and (4), we consider more general solution $y(x, t) = \sum_{n=1}^{\infty} y_n(x, t)$

i.e, $y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}$ ——— (15)

Differentiating (15) partially w.r.t 't', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{E_n\left(-\sin\frac{n\pi ct}{l}\right)\cdot\frac{n\pi c}{l} + F_n \cos\frac{n\pi c}{l}\cdot\frac{n\pi c}{l}\right\}\sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty}\left\{-E_n\frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + \frac{n\pi c F_n}{l}\cos\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}$$ ——— (16)

putting $t=0$ in (15) and (16)

and using the I.C (3) and (4), we get

(15) ⇒ $f(x) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi c(0)}{l} + 0\right\}\sin\frac{n\pi x}{l}$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$ $(\because y(x,0) = f(x))$

and

(16) ⇒ $g(x) = \sum_{n=1}^{\infty}\left\{0 + \frac{n\pi c}{l}F_n \cos\frac{n\pi c(0)}{l}\right\}\sin\frac{n\pi x}{l}$ $\left(\because \frac{\partial y}{\partial t} = g(x) \text{ at } t=0\right)$

$\Rightarrow g(x) = \sum_{n=1}^{\infty}\frac{n\pi c F_n}{l}\sin\frac{n\pi x}{l}$

which are Fourier sine series expansion for $f(x)$ and $g(x)$ respectively in

According we get,

$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ ——— (17)

and $\frac{n\pi c F_n}{l} = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx \Rightarrow F_n = \frac{2}{n\pi c}\int_0^l g(x)\ldots$ (18)

Hence the required solution of given

is $y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\ldots\right.$

where $E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ &

Note:

particular case I : If initial velocity

$y_t(x,0) = g(x) = 0$ then $F_n = 0$ from (18)

∴ In this case the solution (15) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}.$$

where E_n is given by (17)

particular case II : If initial displacement

~~$y(x,0) = f(x) = 0$, then~~ $E_n = 0$ by (16).

∴ In this case the solution (15) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

where F_n is given by (18).

→ A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form $y = y_0 \sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Find the displacement at any point at a distance x from one end at time t.

Sol^n: The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{---(1)}$$

As the end points of the string are fixed, for all time

B.C. $y(0,t) = 0$ and $y(l,t) = 0$ ——(2)

I.C. Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ for $0 \le x \le l$ ——(3)

and initial displacement $= y(x,0) = y_0 \sin\frac{\pi x}{l}$; $0 \le x \le l$ — (4)

proceeding like as in Ex-(1) till equation (15).

i.e. $$y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi c t}{l} + F_n \sin\frac{n\pi c t}{l}\right\}\sin\frac{n\pi x}{l} \quad \text{—— (5)}$$

Differentiating (5) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{-E_n \frac{n\pi c}{l}\sin\frac{n\pi c t}{l} + \frac{n\pi c}{l}F_n \cos\frac{n\pi c t}{l}\right\}\sin\frac{n\pi x}{l} \quad \text{—— (6)}$$

putting $t=0$ in (6) and (7) and using initial conditions (3) & (4) we get

(5) ⇒ $$y(x,0) = y_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \quad \text{—— (7)}$$

(6) ⇒ $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{n=1}^{\infty}\frac{n\pi c}{l}F_n \sin\frac{n\pi x}{l}$$

∴ where $F_n = \frac{2}{n\pi c}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

i.e. $F_n = 0$

From (7), we have

$$y_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

Comparing the coefficients of like terms on both sides, we have

$E_1 = y_0$ and $E_n = 0$ for $n \ne 1$.

Equation (5) reduces to

$$\boxed{y(x,t) = y_0 \sin\frac{\pi x}{l}\cos\frac{\pi c t}{l}}$$

→ for the vibrating string of length π if the ... given by $y(x,0) = C\sin x$, C being constant from rest, then find the displacement $y(x,t)$.

Hint: Taking above example $l = \pi$ and $y_0 = C$

A string of length l has its ends $x=0$ and $x=l$ fixed. It is released from rest in the position $y = \{4\lambda x(l-x)\}/l^2$. Find an expression for the displacement of the string at any subsequent time.

Sol:

The displacement function $y(x,t)$ is the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

subject to boundary conditions:

$$y(0,t) = y(l,t) = 0 \quad \text{for all } t \geq 0 \quad \text{--- (2)}$$

and initial conditions namely

Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ for $0 \leq x \leq l$ --- (3)

Initial displacement $= y(x,0) = f(x) = \frac{4\lambda x(l-x)}{l^2}$ --- (4)

proceeding like as in eq (1) till equation (15)

i.e., $y(x,t) = \sum_{n=1}^{\infty} \left\{ E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l} \right\} \sin\frac{n\pi x}{l}$ --- (5)

Differentiating (5) partially w.r.t 't', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left\{ -\frac{n\pi c}{l} E_n \sin\frac{n\pi ct}{l} + \frac{n\pi c}{l} F_n \cos\frac{n\pi ct}{l} \right\} \sin\frac{n\pi x}{l} \quad \text{--- (6)}$$

putting $t=0$ in eqn (5) and (6) and using initial conditions (3) and (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} = \frac{4\lambda x(l-x)}{l^2}; \ (\text{by } (3)) \quad \text{--- (7)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} F_n \sin\frac{n\pi x}{l} = 0; \ (\text{by } (3))$$

where $F_n = \frac{2}{n\pi c}\int_0^l (0) \sin\frac{n\pi x}{l}\,dx = 0$

and $E_n = \frac{2}{l}\int_0^l \frac{4\lambda x(l-x)}{l^2} \sin\frac{n\pi x}{l}\,dx$

$\Rightarrow E_n = \frac{8\lambda}{l^3}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l}\,dx$

$$= \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \int (l-2x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} + \int 2\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[(0-0) + \left(\frac{2l^3}{n^2\pi^2}\sin n\pi - 0\right) - \frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$= \frac{8\lambda}{l^3}\left[\frac{2l^3}{n^2\pi^2}\sin n\pi - \frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$E_n = \begin{cases} \frac{8\lambda}{l^3}\left[0 - \frac{2l^3}{n^3\pi^3}(1-1)\right] = 0 & \text{if } n = 2m \text{ (even)}, m = 1, 2 \\ \frac{8\lambda}{l^3}\left[0 - \frac{2l^3}{n^3\pi^3}(-1-1)\right] = \frac{32\lambda}{(n\pi)^3} & n = 2m-1 \text{ (odd)}, m = 1, 2 \end{cases}$$

Substituting the values of E_n and F_n expression ①

$$y(x,t) = \frac{32\lambda}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l} \ldots$$

H.W. → A taut string of length l has its ends $x=0$ and $x=l$ fixed. The mid point is taken to a small height h and released from rest at time $t=0$. Find the displacement function $y(x,t)$.–

Hint: B.C. $y(0,t) = y(l,t) = 0 \quad \forall t \geq 0$

Initial position of the string at $t=0$ is made up of two straight line segments OB and BA as shown in the figure and the string is released from rest.

Y, $B(\frac{l}{2}, h)$, h, $O(0,0)$, $M(\frac{l}{2},0)$, $A(l,0)$

The equation of OB is given by

$$y - 0 = \frac{h-0}{(\frac{l}{2}-0)}(x-0) \quad \text{for } 0 \leq x \leq l/2$$

$$\Rightarrow y = \frac{2hx}{l} \quad \text{for } 0 \leq x \leq l/2$$

The equation of BA is given by

$$y - 0 = \frac{h-0}{(\frac{l}{2}-l)}(x-l) \quad \text{for } l/2 \leq x \leq l.$$

$$\Rightarrow y = \frac{2h(l-x)}{l} \quad \text{for } \frac{l}{2} \leq x \leq l.$$

Hence, the initial displacement is given by

$$y(x,0) = f(x) = \begin{cases} 2hx/l, & 0 \leq x \leq l/2 \\ \frac{2h(l-x)}{l}, & l/2 \leq x \leq l. \end{cases}$$

and the initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

Ans: $y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$

where $E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2hx}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2h(l-x)}{l} \sin\frac{n\pi x}{l}\,dx\right]$$

$$= \begin{cases} \frac{8(-1)^{m+1} h}{(2m-1)^2\pi^2}, & \text{if } n = 2m-1 \text{ (odd)} \quad m = 1,2,\ldots \\ 0, & \text{if } n = 2m \text{ (even)} \quad m = 1,2,\ldots \end{cases}$$

$$\therefore y(x,t) = \frac{8h}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2} \sin\frac{(2m-1)\pi x}{l} \cos\frac{(2m-1)\pi ct}{l}$$

→ A tightly stretched elastic string of length l, with fixed end points $x=0$ and $x=l$ is initially in the position is given by $y = y_0 \sin^3\frac{\pi x}{l}$, y_0 being constant. Find the displacement $y(x,t)$.

Hint: B.C. $y(0,t) = y(\pi,t) = 0$, $\forall t \geq 0$.

I.C. Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ for $0 \leq x \leq \pi$ — ①

Initial displacement $= y(x,0) = y_0 \sin^3\frac{\pi x}{l}$ — ②

proceeding like as in in ex-①, till eqn ⑮,

we have

$$y(x,t) = \sum_{n=1}^{\infty} \left\{E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right\} \sin\frac{n\pi x}{l} \quad \text{— ③}$$

Differentiating ③ partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left\{-\frac{n\pi c}{l} E_n \sin\frac{n\pi ct}{l} + \frac{n\pi c}{l} F_n \cos\frac{n\pi ct}{l}\right\} \sin\frac{n\pi x}{l} \quad \text{— ④}$$

putting $t=0$ in ③ and ④

and using the I.C. ① and ②, we get

$$③ \Rightarrow y(x,0) = y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \quad \text{— ⑤}$$

$$④ \Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} F_n \sin\frac{n\pi x}{l} \quad \text{— ⑥}$$

$$\text{where } F_n = \frac{2}{n\pi c} \int_0^l (0) \sin\frac{n\pi x}{l} dx = 0$$

now from ⑤

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = E_1 \sin\frac{\pi x}{l} + \ldots$$

comparing the coefficient

we have $E_1 = \frac{3y_0}{4}$, $E_2 = 0$, $E_3 =$

Substituting these values in (5),
the required displacement is given by

$$y(x,t) = \frac{3y_0}{4} \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l} - \frac{y_0}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi ct}{l}$$

→ A tightly stretched elastic string of length π, with fixed end points $x=0$ and $x=\pi$ is initially in the position is given by $y = y_0 \sin^3 x$, y_0 being constant. Find the displacement $y(x,t)$.

Ans: $y(x,t) = \frac{3y_0}{4} \sin x \cos ct - \frac{y_0}{4} \sin 3x \cos 3ct$

[putting $l=\pi$ in the above problem]

→ Solve the one dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$, $0 \le x \le 2\pi$, $t \ge 0$ subject to the following initial and boundary conditions.

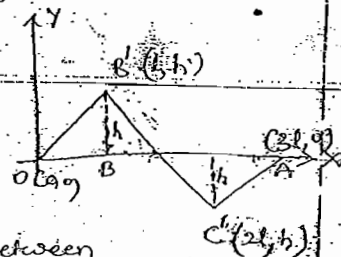
(i) $y(x,0) = \sin^3 x$, $0 \le x \le 2\pi$ (ii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, $0 \le x \le 2\pi$

(iii) $y(0,t) = y(2\pi, t) = 0$, for $t \ge 0$.

→ Find the deflection $y(x,t)$ of the vibrating string (length $=\pi$, and $c^2=1$) corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$.

Ans: $k(\cos t \sin x - \cos 2t \sin 2x)$

2003 → The points of trisection of a string are pulled aside through the same distance h on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.

(or)

Find the deflection $u(x,t)$ of vibrating string, stretched between fixed points $(0,0)$ and $(3l, 0)$, corresponding to zero initial velocity and following initial deflection

$$f(x) = \begin{cases} \frac{hx}{l} & \text{when } 0 \le x \le l \\ \frac{h(3l-2x)}{l} & \text{when } l \le x \le 2l \\ \frac{h(x-3l)}{l} & \text{when } 2l \le x \le 3l \end{cases} \quad \text{— (A)}$$

where h is constant.

Sol^n: The displacement $y(x,t)$ of any point of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

B.C $y(0,t) = y(3l, t) = 0 \quad \forall t \ge 0$ — (1)

I.C $y(x,0) = f(x)$ — (2) (where $f(x)$ is given by (A))

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ — (3)

Proceeding like as in Ex-(1) till eqn (15) by replacing l by $3l$,

we have,

$$y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi ct}{3l} + F_n \sin\frac{n\pi ct}{3l}\right\}\sin\frac{n\pi x}{3l} \quad \text{--- (4)}$$

Differentiating (3) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{-E_n\frac{n\pi c}{3l}\sin\frac{n\pi ct}{3l} + \frac{n\pi c}{3l}F_n\cos\frac{n\pi ct}{3l}\right\}\sin\frac{n\pi x}{3l} \quad \text{--- (5)}$$

putting $t=0$ in (4) and (5)

and using the B.C. (2) and (3), we get

(5) $\Rightarrow$ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} F_n \frac{n\pi c}{3l}\sin\frac{n\pi x}{3l} = 0$ (by (3))

where $F_n = \frac{2}{n\pi c}\int_0^{3l}(0)\sin\frac{n\pi x}{3l}\,dx = 0$

(4) $\Rightarrow$ $y(x,0) = f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{3l}$

where $E_n = \frac{2}{3l}\int_0^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx$. --- (6)

Now

$$E_n = \frac{2}{3l}\int_0^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx$$

$$= \frac{2}{3l}\left[\int_0^{l} f(x)\sin\frac{n\pi x}{3l}\,dx + \int_l^{2l} f(x)\sin\frac{n\pi x}{3l}\,dx + \int_{2l}^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx\right]$$

$$= \frac{2}{3l}\left[\int_0^{l}\frac{hx}{l}\sin\frac{n\pi x}{3l}\,dx + \int_l^{2l}\frac{h(3l-2x)}{l}\sin\frac{n\pi x}{3l}\,dx + \int_{2l}^{3l}\frac{h(x-3l)}{l}\sin\frac{n\pi x}{3l}\,dx\right]$$

Continuing in this way we get

$$E_n = \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left\{\sin\left(n\pi - \frac{n\pi}{3}\right)\right\}\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left(\sin n\pi\cos\frac{n\pi}{3} - \cos n\pi\sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - 0 + \cos n\pi \sin\frac{n\pi}{3}\right] \qquad \left(2\sin\frac{n\pi}{2} = 0\right)$$

$$= \frac{18h}{n^2\pi^2}\left[1+\cos n\pi\right]\sin\frac{n\pi}{3}$$

$$= \frac{18h}{n^2\pi^2}\left[1+(-1)^n\right]\sin\frac{n\pi}{3}$$

Thus $E_n = 0$ if n is odd.

$$E_n = \frac{36h}{n^2\pi^2}\sin\frac{n\pi}{3} \quad \text{if } n \text{ is even}$$

put $n = 2m$, $m = 1, 2 \ldots$

i.e., $$= \frac{36h}{4m^2\pi^2}\sin\frac{2m\pi}{3}, \quad m = 1, 2, \ldots$$

$$= \frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}$$

putting the values of E_n and F_n in (4), the required deflection is given by

$$y(x,t) = \sum_{m=1}^{\infty}\frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}\sin\frac{n\pi x}{3l}\cos\frac{n\pi ct}{3l}$$

$$\Rightarrow y(x,t) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\sin\frac{2m\pi}{3}\cos\frac{n\pi ct}{3l}\sin\frac{n\pi x}{3l} \quad (7)$$

putting $x = \frac{3l}{2}$ in (7); we find that the displacement of the midpoint of the string

i.e. $y\left(\frac{3l}{2}, t\right) = 0$

because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string always rest.

A tightly stretched string of length ... ends is initially in equilibrium posit... vibrating by giving each point a vel... Find displacement. Ans: $y(x,t) = \frac{l v_0}{12\pi c}[$

2004 → A uniform string of length l held tightly between $x=0$ and $x=l$ with no initial displacement, is struck at $x=0$, $0<a<l$ with velocity v_0. Find the displacement of the string at any time $t>0$.

Hint: B.C. $y(0,t)=y(l,t)=0 \quad \forall t$

I.C. initial displacement $y(x,0)=0, \quad 0\le x\le l$.

Initial velocity $= y_t(x,0) = v_0 \; ; \; 0\le x\le l$.

Ans: $y(x,t) = \frac{4v_0 l}{c\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$

2000 A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $kx(l-x)$, find its displacement.

Ans: $y(x,t) = \frac{8kl^3}{c\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$

2006 → A deflection of a vibrating string of length l, is governed by the partial differential equation $y_{tt}=c^2y_{xx}$. The ends of the string are fixed at $x=0$ and l. The initial velocity is zero. The initial displacement is given by

$$y(x,0)=\begin{cases}\frac{x}{l}, & 0<x<l/2\\ \frac{l-x}{l}, & l/2<x<l\end{cases}$$

Find the deflection of the string at any instant of time.

Ans: $y(x,t) = \frac{4l}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$

2009, IFS-2005

→ A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$ the string is given a shape defined by $f(x)=\mu x(l-x)$ where μ is constant and then released. Find the displacement of any point x of the string at any time $t>0$.

Ans: $y(x,t) = \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{l} \cos\frac{(2n-1)\pi ct}{l}$

→ A string of length l is initially at rest in its equilibrium position and motion is started giving each of its points a velocity is given by $v=kx$ if $0 \le x \le l/2$ and $v=k(l-x)$ if $\frac{l}{2} \le x \le l$. Find the displacement function $y(x,t)$.

Ans: $\frac{4kl^2}{c\pi^3} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{l} \sin\frac{(2m-1)\pi ct}{l}$

→ If the string of length l is initially at rest in equilibrium position and each of its points is given the velocity $v_0 \sin\frac{3\pi x}{l} \cos\frac{2\pi x}{l}$ where $0<x<l$ at $t=0$. Find the displacement function.

Ans: $y(x,t) = \frac{l v_0}{2\pi c} \sin\frac{\pi x}{l} \sin\frac{\pi ct}{l} + \frac{l v_0}{5\pi c} \sin\frac{5\pi x}{l} \sin\frac{5\pi ct}{l}$

→ A string is stretched between the fixed points $(0,0)$ and $(l,0)$ and released at rest from the given by $f(x) = \begin{cases} \frac{2kx}{l}, & \text{when } 0<x<l/2 \\ \frac{2k(l-x)}{l}, & \text{when } l/2<x<l \end{cases}$

Find that deflection of the string at

Ans: $\frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos\frac{(2n-1)\pi ct}{l} \sin\frac{(2n-1)\pi x}{l}$

A taut string of length 20 cms. fastened at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity is given by

$v = x$ in $0 \le x \le 10$

$= 20 - x$ in $10 \le x < 20$, x being the distance from one end. Determine the displacement at any subsequent time.

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MATHEMATICS By K. VENKANNA

* Numerical Analysis *

Solution of Algebraic and Transcendental Equations:

Introduction: In this chapter, we shall discuss some numerical methods for solving algebraic and transcendental equations.

The equation $f(x)=0$ is said to be algebraic if $f(x)$ is purely a polynomial in x. If $f(x)$ contains some other functions, namely, Trigonometric, Logarithmic, Exponential, etc., then equation $f(x)=0$ is called a Transcendental Equation.

The equations $x^3-7x+8=0$ and $x^4+4x^3+7x^2+6x+3=0$ are algebraic.

The equations $3\tan 3x = 3x+1$, $x-2\sin x=0$ and $e^x=4x$ are transcendental.

→ Algebraically, the real number α is called the real root (or zero) ~~function $f(x)$ of the equa~~ if and only if $f(\alpha)=0$ and the real root of an eq

is the value of x where the graph of $f(x)$ meets the x-axis in rectangular co-ordinate system.

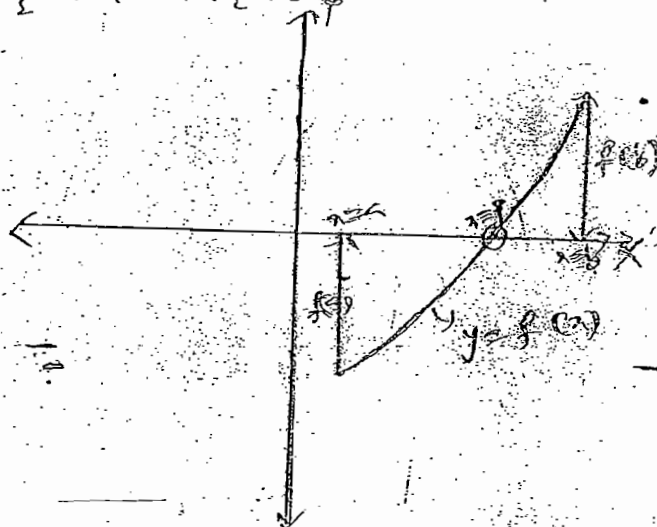
We shall assume that the equation $f(x)=0$ —— (1) has only isolated roots, that is for each root of the equation there is a neighbourhood which does not contain any other roots of the equation.

Approximately the isolated roots of the equation (1) has two stages.

(1) Isolating the roots that is finding the smallest possible interval (a,b) containing one and only one root of the equation (1)

(2) Improving the values of the approximate roots to the specified degree of accuracy. Now we state a very useful theorem of mathematical analysis without proof.

Theorem: Intermediate value property:
If $f(x)$ is a real valued continuous in the closed interval $a \le x \le b$. If $f(a)$ and $f(b)$ have opposite signs,

then the graph of the function $y = f(x)$ crosses the x-axis at least once; that is $f(x) = 0$ has at least one root ξ s.t $a < \xi < b$.

✓ Broadly speaking, all the known numerical methods for solving either a transcendental equation (or) an algebraic equation can be classified into two groups: direct methods and iterative methods.

Direct methods require no knowledge of the initial approximation of a root of the equation $f(x) = 0$.

Iterative methods do req
approximation to initi
(Iteration means repeated appli
process or a pattern of ac

— How to get the first approximation?
We can find the approximate value of the root of $f(x) = 0$ either by a graphical method (or) by an analytical method as explained below:

Graphical method:

The real root of the equation — $f(x) = 0$ ——— (1) can be determined approximately as the abscissas of the points of intersection of the graph of the function $y = f(x)$ with the x-axis. If $f(x)$ is simple, we shall draw the graph of $y = f(x)$ w.r.t a rectangular axis x'ox and y'oy. The points at which the graph meets the x-axis are the location of the roots of (1).
If $f(x)$ is not simple we replace equation (1) by an equivalent equation say $\phi(x) = \psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ are simpler than $f(x)$. Then the x-co-ordinate of the point of intersection of the graphs gives the crude approximation of the real roots of the equation (1).

problem (1)

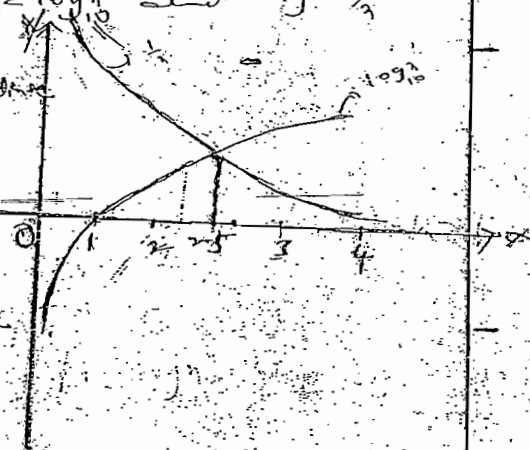
Solve the equation $x \log_{10} x = 1$, graphically. —(1)

Sol: The given equation (1) $x\log_{10} x = 1$ can be written

as $\log_{10} x = 1/x$ —(2)

where $\log_{10} x$ and $\frac{1}{x}$

simpler than $x\log_{10} x = 1$, constructing the curves $y = \log_{10} x$ and $y = 1/x$

we get x-coordinate of the point of intersection as 2.5.

∴ The approximate value of the root of $x\log_{10} x = 1$ is $\xi = 2.5$

HW → Solve $x^2 + x - 1 = 0$ graphically

HW → Solve $e^{2x} + x + 0.1 = 0$ ...

→ Solve $x - \sin x - 1 = 0$, ...

Sol: Let the given equ ...

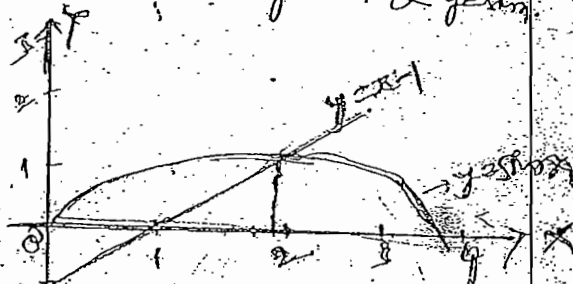
$f(x) = x - \sin x$ ...

It can be written as

$$x - 1 = \sin x$$

where $x-1$ and $\sin x$ simpler than (1).

constructing the curves $y = x-1$ & $y = \sin x$

we get x-co-ordinate of the point of the intersection as 1.9

∴ The approximate value of the root of (1) is $\xi = 1.9$.

* Analytical method

This method is based on 'intermediate value property'. We shall illustrate it through an example.

Let $f(x) = 3x - \sqrt{1 + \sin x} = 0$ — (1)

now $f(0) = -1$,

$f(1) = 3 - \sqrt{1 + \sin(1)}$

$= 3 - \sqrt{1 + 0.84147}$

$= 1.64299$

∴ $f(0) < 0$ & $f(1) > 0$

i.e $f(0)$ and $f(1)$ are opposite signs.

∴ By intermediate value property there is atleast one root b/w $x=0$ and $x=1$.

This method is often used to find the first approximation to a root of either transcendental equation or algebraic eqn. Hence, in analytical method, we must always start with an initial interval, (a,b), so that f(a) and f(b) have opposite signs.

* Bisection Method :—

This method is due to Bolzano.

Suppose, we wish to locate the root of an equation $f(x)=0$ in an interval, say, (x_0, x_1). Let $f(x_0)$ and $f(x_1)$ are of opposite signs, such that $f(x_0) \cdot f(x_1) < 0$.

Then the graph of the function crosses the x-axis between x_0 and x_1, which guarantees the existence of atleast one root in the interval (x_0, x_1).

The desired root is approximately defined by the midpoint

If $f(x_2) = 0$, then x_2 is the desired root of $f(x) = 0$. However, if $f(x_2) \neq 0$, then the root may be between x_0 and x_2 (or) x_2 and x_1.

Now, we define the next approximation by $x_3 = \frac{x_0 + x_2}{2}$ provided $f(x_0) \cdot f(x_2) < 0$, then the root may be found b/w x_0 and x_2 (or) by $x_3 = \frac{x_1 + x_2}{2}$ provided $f(x_1) f(x_2) < 0$, then the root lies b/w x_1 and x_2 etc.

Thus, at each step, we either find the desired root to the required accuracy or narrow the range to half the previous interval as depicted in the given figure. This process of halving the intervals is continued to determine a small and smaller interval within which the desired root lies. Continuation of this process eventually gives us the desired root.

(This method is known as an Iteration Method)

Geometrical illustration of bisection method.

→ Solve $x^3 - 9x + 1 = 0$ for the root b/w $x = 2$ and $x = 4$ by the bisection method.

Sol Let $f(x) = x^3 - 9x + 1$.

Since $f(2) = -9 < 0$, $f(4) = 29 > 0$

$\therefore f(2) \cdot f(4) < 0$

Hence the root lies b/w 2 and 4.

Let $x_0 = 2$, $x_1 = 4$. Then the first approximation to the root is $x_2 = \frac{x_0 + x_1}{2}$

$= \frac{2+4}{2} = 3 \Rightarrow \boxed{x_2 = 3}$

Since $f(x_2) = f(3) = 1 (> 0)$

$\therefore f(2) \cdot f(3) < 0$ i.e $f(x_0) \cdot f(x_2) < 0$

Hence the root lies b/w 2 and 3

the second approximation to the root is $x_3 = \frac{x_0 + x_2}{2} = \frac{2+3}{2} = \frac{5}{2} = 2.5$

$\therefore \boxed{x_3 = 2.5}$

Since $f(x_3) = f(2.5) < 0$

$\therefore f(x_2) \cdot f(x_3) < 0$

i.e $f(3) \cdot f(2.5) < 0$.

Hence the root lies b/w 3 & 2.5,

the third approximation to the root is $x_4 = \frac{x_2 + x_3}{2} = \frac{3 + 2.5}{2} = 2.75$

i.e $\boxed{x_4 = 2.75}$

Sly, we can find that $\boxed{x_5 = 2.875}$ and $x_6 =$

and the process can be co[ntinued] until the root is obtained to desired accuracy.

Now, we can write in

n	x_n	$f(x_n)$
2	3	1.0
3	2.5	−5.875
4	2.75	−2.9531
5	2.875	−1.1113
6	2.9375	−0.0901.

→ Solve the equation $x^3 - 9x + 1$ for the root lying b/w 2 and 3, correct to three significant figures.

Sol. Let $f(x) = x^3 - 9x + 1$

Since $f(2) = 8 - 18 + 1$
$= -9 < 0$ and

$f(3) = 27 - 27 + 1$
$= 1 > 0$.

$\therefore f(2) \cdot f(3) < 0$.

Hence the root lies b/w 2 & 3.

Let $a_0 = 2$, $b_0 = 3$

n	a_n (−ve)	b_n (+ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	−5.8 (<0)
1	2.5	3	2.75	−2.9 (<0)
2	2.75	3	2.88	−1.03 (<0)
3	2.88	3	2.94	−0.05 (<0)
4	2.94	3	2.97	0.47 (>0)
5	2.94	2.97	2.955	0.21 (>0)
6	2.94	2.955	2.9475	0.08 (>0)
7	2.94	2.9475	2.9438	0.017 (>0)
8	2.94	2.9438	2.9419	−0.016 (<0)
[illegible]	2.9419	2.9438	2.9428	0.00[illegible] (>0)

In the 8th step a_n, b_n and x_{n+1} are equal upto three significant figures.

(b)

We can take 2.94 as a root upto three significant figures.

$\therefore$ The root of $x^3 - 9x + 1 = 0$ is 2.94

→ Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method in four stages.

Sol Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is $-ve$ and $f(3)$ is $+ve$

$\therefore f(2)\, f(3) < 0$

Hence the root lies b/w 2 and 3.

$\therefore$ first approximation to the root

is $x_1 = \frac{2+3}{2} = \boxed{2.5}$

Now $f(x_1) = f(2.5)$

$= (2.5)^3 - 4(2.5) - 9$

$= -3.375$

$= -ve$

$\therefore f(3) \cdot f(x_1) < 0$

Hence the root lies b/w x_1 and 3.

$\therefore$ second approximation to the root is $x_2 = \frac{x_1 + 3}{2}$

$= \frac{2.5+3}{2}$

$= \boxed{2.75}$

Now $f(x_2) = f(2.75)$

$= (2.75)^3 - 4(2.75) - 9$

$= 0.7969$

ie $+ve$

$\therefore f(x_1)\, f(x_2) < 0$

Hence the root lies b/w x_1 ...

The third approximation to the root is $x_3 = \frac{x_1+x_2}{2}$

$= \boxed{2.625}$

Now $f(x_3) = (2.625)^3 - 4(2.625) - 9$

$= -1.4121$

i.e. $-ve$

$\therefore f(x_2) f(x_3) < 0$

Hence the root lies b/w x_2 and x_3.

$\therefore$ The fourth approximation to the root is $x_4 = \frac{1}{2}(x_2+x_3)$

$= \boxed{2.6875}$

Hence the root is 2.6875 approximately.

→ Find the real root to four decimals of the equation $x^6 - x^4 - x^3 - 1 = 0$ which lies b/w 1 and 2.

Sol Let $f(x) = x^6 - x^4 - x^3 - 1$

Since $f(1) = -2 < 0$ &

$f(2) = 39 > 0$

$\therefore f(1) \cdot f(2) < 0$.

Hence the root lies b/w 1 & 2

The first approximation to the root is $x_1 = \frac{1+2}{2} = \boxed{1.5}$

Now $f(x_1) = f(1.5)$

$= +ve$

$\therefore f(1) \cdot f(x_1) < 0$

Hence the root lies b/w 1 & x_1

The second approximation to the root is $x_2 = \frac{1+x_1}{2}$ $\boxed{1.25}$

Now $f(x_2) = f(1.25)$
$= -ve$.

$\therefore f(x_1) \cdot f(x_2) < 0$.

Hence the root lies b/w x_2 & x_1.

The third approximation to the root

is $x_3 = \frac{x_2 + x_1}{2}$

$= \frac{1.25 + 1.5}{2} = \boxed{1.375}$

Now $f(1.375)$ is $+ve$

$\therefore f(x_3) \cdot f(x_1) < 0$.

Hence the root lies b/w x_3 & x_1.

The fourth approximation to the root is $x_4 = \frac{x_3 + x_1}{2}$

$= \frac{1.375 + 1.5}{2}$

$= \boxed{1.4375}$

Now $f(x_4) = f(1.4375)$

$= +ve$

$\therefore f(x_3) \cdot f(x_4) < 0$

Hence the root lies b/w x_3 & x_4.

Now the fifth approximation to the root is $x_5 = \frac{x_3 + x_4}{2}$

$= 1.3775$

$= \boxed{1.4 \ldots}$

Now $f(x_5) = +ve$.

$\therefore f(x_1) \cdot f(x_5) < 0$

Hence the root lies b/w ...

The ... approximation

is $x_6 = \frac{x_3 + x_5}{2}$

$= \frac{1.375 + 1.40625}{2}$

$= \boxed{1.390625}$

Now $f(x_6) = -ve$

$\therefore f(x_5)\, f(x_6) < 0$

Hence the root lies b/w x_6 & x_5

The 7^{th} approximation to the root is $x_7 = \frac{x_6 + x_5}{2}$

$= \frac{1.390625 + 1.40625}{2}$

$= \boxed{1.3984375}$

Now $f(x_7) = f(1.3984375)$

$= -ve.$

$\therefore f(x_7) \cdot f(x_5) < 0$

Hence the root lies b/w x_7 & x_5

The 8^{th} approximation to the root is $x_8 = \frac{x_7 + x_5}{2}$

$= \frac{1.3984375 + 1.40625}{2}$

$= \boxed{1.40234375}$

Now $f(x_8) < 0$

$\therefore x_9 = \frac{x_8 + x_5}{2}$

$= \frac{1.40234375 + 1.40625}{2}$

$= 1.4043$ (nearly)

Now $f(x_9) > 0$

$$\therefore x_{10} = \frac{x_8 + x_9}{2}$$

$$= \frac{1.40234375 + 1.4043}{2}$$

$$= \boxed{1.4033}$$

Now $f(x_{10}) < 0$

$$\therefore x_{11} = \frac{x_{10} + x_9}{2}$$

$$= \frac{1.4033 + 1.4043}{2}$$

$$= \boxed{1.4038}$$

Now $f(x_{11}) =$ +ve.

$$\therefore x_{12} = \frac{x_{10} + x_{11}}{2}$$

$$= \frac{1.4033 + 1.4038}{2}$$

$$= \boxed{1.40355}$$

Hence the root to four decimals of $x^6 - x^4 - x^3 - 1 = 0$ lying b/w 1 and 2 is 1.4036 (approximately).

HW → Find to three decimals a root of the equation $3x - \cos x - 1 = 0$.

[Ans: 0.607]

HW → Compute one root of $e^x - 3x = 0$ correct to two decimal places.

[Ans: 1.51]

HW → Find the root of $\tan x + x = 0$ upto two decimal places which lies b/w 2 and 2.1

Ans: 2.03

HW → Find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places by using bisection method.

Ans: 0.7065

HW → Compute one +ve root of $2x - 3\sin x - 5 = 0$, by bisection method, correct to three significant figures.

Ans: 2.88

HW → Compute one root of $x + \log x - 2 = 0$ correct to two decimal places which lies b/w 1 and 2

Ans: 1.56

Note :① While applying bisection method we must be careful to check that $f(x)$ is continuous.

For example, we may come across functions like $f(x) = \frac{1}{x-1}$. If we consider the interval $(0.5, 1.5)$, then $f(.5)\, f(1.5) < 0$. In this case we may be tempted to use bisection method. But we cannot use the method here because $f(x)$ is not defined at middle point $x=1$. We can overcome these difficulties by taking $f(x)$ to be continuous throughout the initial bisecting interval (Note that, if f is a continuous function on $[a,b]$ and $f(a) \neq f(b)$ then f assumes every value b/w $f(a)$ and $f(b)$.)

Therefore we should always examine the continuity of the function in the initial interval before attempting the bisection method.

Note(2): It may happen that a function has more than one root in an interval. The bisection method helps us in determining one root only. We can determine the other roots by properly choosing the initial intervals.

— A numerical process starts with an initial approximation and iterati[on] [of] this approximation until we get the accurate value of the root.

Let us consider another iter[ation] method now:

* Regula Falsi Method :-

— This method is also known as the method of false position.

— The Latin word Regula Falsi means rule of falsehood. It doesnot mean that the rule is a false statement. But it conveys that the roots that we get according to the rule are approximate roots and not necessarily exact roots. This method is similar to the bisection method.

— The bisection method for finding approximate roots has a drawback that it makes use of only the signs of f(a) and f(b). It doesnot use the values f(a), f(b) in the computations.

for example -
If $f(a) = 700$ and $f(b) = -0.1$, then by the bisection method the first approximate value of a root of $f(x)$ is the mid value x_0 of the interval (a, b). But at x_0, $f(x_0)$ is nowhere near 0.

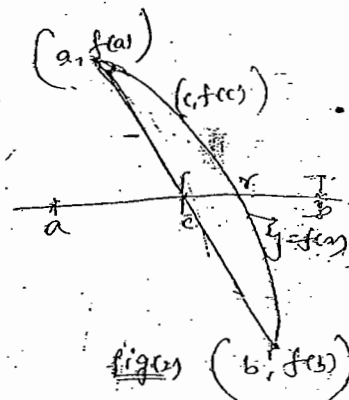
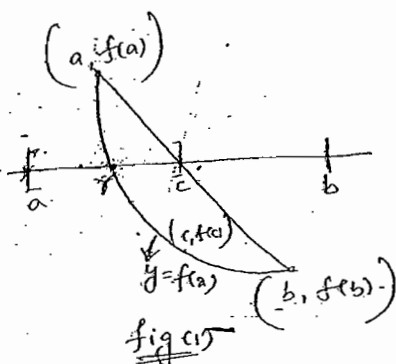
∴ In this case it makes more sense to take a value near to −0.1 than the middle value as the approximation to the root.

This drawback is to some extent overcome by the Regula-falsi method.

Geometrically, suppose we want to find a root of the eqn $f(x)=0$ where $f(x)$ is a continuous function. As in the bisection-method, we first find an interval (a,b) such that $f(a)f(b)<0$.

fig (1)

(a, f(a)) (c, f(c)) a c r b y=f(x) (b, f(b))

fig (2)

The condition $f(a)f(b)<0$ means that the points $(a, f(a))$ and $(b, f(b))$ lie on the opposite sides of the x-axis.

The line joining $(a, f(a))$ and $(b, f(b))$ crosses the x-axis at some point $(c, 0)$. Then we take the x-coordinate of that point as the first approximation.

If $f(c)=0$, then $x=c$ is the required root.

If $f(a)f(c)<0$, then the root lies in (a,c) (fig(1)). In this case the graph of $y=f(x)$ is concave near the root r. Otherwise if $f(a)f(c)>0$, the root lies in (c,b) (fig(2)) & this case the graph of $y=f(x)$ is convex.

Having fixed the interval in which root lies, we repeat the above pr

In mathematical form,

The formula for the line joining the two points $(a, f(a))$ and $(b, f(b))$ is given by

$$y - f(a) = \frac{f(b) - f(a)}{b-a}(x-a).$$

$$\Rightarrow \frac{y - f(a)}{f(b) - f(a)} = \frac{x-a}{b-a} \quad \text{——— (1)}$$

Since the straight line intersects the x-axis at $(c, 0)$, the point $(c, 0)$ lies on the straight-line. putting $x=c$, $y=0$ in eqn (1), we get

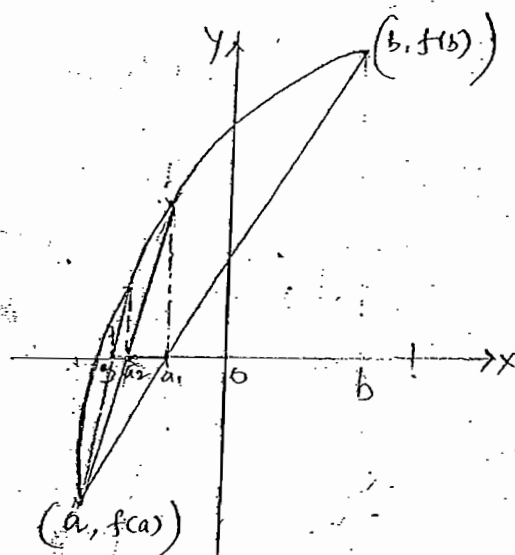
$$\frac{-f(a)}{f(b) - f(a)} = \frac{c-a}{b-a}$$

$$\Rightarrow \frac{c}{b-a} - \frac{a}{b-a} = \frac{-f(a)}{f(b) - f(a)}$$

$$\Rightarrow c = a - \frac{f(a)}{f(b) - f(a)}(b-a)$$

$$\Rightarrow c = \frac{a f(b) - b f(a)}{f(b) - f(a)} \quad \text{——— (2)}$$

This expression for 'c' gives an approximate value of a root of $f(x)$.

Now, examine the sign of $f(c)$ and decide in which interval (a, c) or (c, b) the root lies. we thus obtain a new interval such that $f(x)$ is of opposite signs at the end points of this interval. By repeating this process, we get a sequence of intervals (a, b), (a, a_1), (a, a_2), ...

We stop the process when either of the following holds.

(i) The interval containing the zero of $f(x)$ is of sufficiently small length

(or)

(ii) The difference between two successive approximations is negligible.

In the iteration format, the method is usually written as

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{——— } \boxed{I}$$

where (x_0, x_1) is the interval in which the root lies.

→ We now summarise this method form:

Step 1: Find numbers x_0 and x_1 such

Step 2: Set $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$.

This gives the first app

Step 3: If $f(x_2)=0$ then x_2 is the required root. If $f(x_2)\neq 0$ and $f(x_0)f(x_2)<0$, then the next approximation lies in (x_0, x_2). Otherwise it lies in (x_2, x_1).

Step 4: Repeat the process till the magnitude of the difference between two successive iterated values x_i and x_{i+1} is less than the accuracy required.

Note: $|x_{i+1}-x_i|$ gives the error after i^{th} iteration.

2003 Find a real root of the eqn $x^3-2x-5=0$ by the method of false position to three decimal places.

Sol^n: Let $f(x)=x^3-2x-5$.

So that $f(2)=-1$ and $f(3)=16$

$\therefore f(2)\,f(3)<0$

Hence ~~the root~~ lies b/w 2 and 3.

Take $x_0=2$, $x_1=3$.

$f(x_0)=-1$, $f(x_1)=16$

By the method of false position, we get

$$x_2 = x_0 - \frac{(x_1-x_0)}{f(x_1)-f(x_0)} f(x_0) \quad \text{---} (1)$$

$$= 2 - \frac{3-2}{16+1}(-1)$$

$$= 2+\frac{1}{17} = \frac{35}{17} = 2.0588$$

Now $f(x_2) = -0.3908$.

$\therefore f(2.0588)\cdot f(3)<0$

Hence the root lies between 2.0588 and 3.

Take $x_0 = 2.0588$, $x_1 = 3$

$\therefore f(x_0) = -0.3908$, $f(x_1) = 16$.

from (1)

$$x_3 = 2.0588 - \frac{3 - 2.0588}{16 + 0.3908}(-0.3908)$$

$= 2.0813$.

Now repeating this process, the successive approximations are given by

$x_4 = 2.0862$, $x_5 = 2.0915$, $x_6 = 2.0934$,

$x_7 = 2.0941$, $x_8 = 2.0943$ etc.

$\therefore$ The approximate root is 2.094 correct to 3 decimal places.

→ The equation $x^3 + 7x^2 + 9 = 0$ has a root b/w -8 and -7. Use the Regula-Falsi method to obtain the root rounded to 3 decimal places. Stop the iteration when $|x_{i+1} - x_i| < 10^{-4}$.

Sol^n: Let $f(x) = x^3 + 7x^2 + 9$.

Take $x_0 = -8$ and $x_1 = -7$.

$f(x_0) = f(-8) = -55 < 0$

$f(x_1) = f(-7) = 9 > 0$.

By method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{--- (1)}$$

$$= \frac{(-8)(9) - (-7)(-55)}{9 + 55}$$

$x_2 = -7.1406$.

$\therefore$ The first approximation to

Now $f(x_2) = 1.862856 > 0$

and $f(x_0) f(x_2) = f(-8) f(-7.1406) < 0$

Hence the root lies between -8 and -7.1406.

Take $x_0 = -8$ and $x_1 = -7.1406$.

$\therefore f(x_0) = -55$ and $f(x_1) = 1.862856$

$\therefore$ from (1)

$$x_3 = \frac{(-8)(1.862856) + (7.1406)(-55)}{1.862856 + 55}$$

$$= -7.168174.$$

$\therefore$ The second approximation to the root is $x_3 = -7.168174$.

Now repeating this process, the successive approximations are given by

$x_4 = -7.1735649$, $x_5 = -7.1745906$

$x_6 = -7.1747855$, $x_7 = -7.1748226$.

The absolute value of the difference between the 6^{th} and 7^{th} iterated values is $|7.1748226 - 7.1747855| = 0.0000371 < 10^{-4}$.

$\therefore$ we stop the iteration here.

Further, the value of $f(x)$ at 6^{th} iterated value is $0.0004697 8 = 4.6978 \times 10^{-4}$ which is close to zero.

Hence -7.175 is an approximate root of $x^3 + 7x^2 + 9 = 0$ rounded-off to 3 decimal places.

(8) -2008 Determine an approximate root of the equation $\cos x - xe^x = 0$ using Regula-Falsi method, correct to 4 decimal places (13)

Sol^n^: $f(x) = \cos x - xe^x$

So that $f(0) = 1$ and $f(1) = \cos 1 - e$
$= -2.17798$

$\therefore f(0)\,f(1) < 0$

Hence, the root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$.

$\therefore f(x_0) = 1$ and $f(x_1) = -2.17798$

By the method of false position, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{--- (1)}$$

$$= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1}$$

$$= 0.31467$$

$\therefore$ The first approximation to the root is

$x_2 = 0.31467$

Now $f(x_2) = 0.51987 > 0$

$\therefore f(x_2)\,f(x_1) < 0$

$\therefore$ The root lies b/w 0.31467 and 1

Take $x_0 = 0.31467$ and $x_1 = 1$

$\therefore f(x_0) = 0.51987$ and $f(x_1) = -2.17798$

From (1),

$$x_3 = \frac{(0.3146)(-2.17798)}{-2.17798}$$

$x_3 = 0.44673$

The 2nd approximation to

$x_3 = 0.446$

Now repeating this process, the successive approximations are

$x_4 = 0.49402$, $x_5 = 0.50995$,

$x_6 = 0.51520$, $x_7 = 0.51692$; $x_8 = 0.51748$.

$x_9 = 0.51767$, $x_{10} = 0.51775$, etc.

$\therefore$ The approximate root is 0.5177 correct to 4 decimal places

→ Find a real root of the eqn $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places. **Ans: 2.7406**

→ Use the method of false position to find the fourth root of 32 correct to three decimal places.

Sol^n: Let $x = (32)^{1/4}$ then $x^4 = 32 \Rightarrow x^4 - 32 = 0$

Let $f(x) = x^4 - 32$. **Ans: 2.378**

2007, 12 pt → Use the method of false-position to find a real root of $x^3 - 5x - 7 = 0$ lying between 2 and 3 correct to 3 places of decimals.

→ Use the Regula-Falsi method to compute a real root of the eqn $x^3 - 9x + 1 = 0$

(i) if the root lies b/w 2 and 4

(ii) if the root lies b/w 2 and 3.

Comment on the results.

→ Use Regula-Falsi Method to find a real root of the eqn $\log x - \cos x = 0$. accurate to four decimal places after three successive approximations. **Ans: 1.3030**

1998 → Use Regula-Falsi method to show that the real root of $x \log_{10} x = 1.2$ lies b/w 3 and 2.740646.

Note: In regula-falsi method, at each stage we find an interval (x_0, x_1) which contains a root and then apply iteration formula [I]. This procedure has a disadvantage. To overcome this, regula-falsi method is modified.

The modified method is known as Secant method.

In this method we choose x_0 and x_1 as any two approximations of the root. The interval (x_0, x_1) need not contain the root. Then we apply formula [I] with $x_0, x_1, f(x_0)$ and $f(x_1)$.

The iterations are now defined as:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

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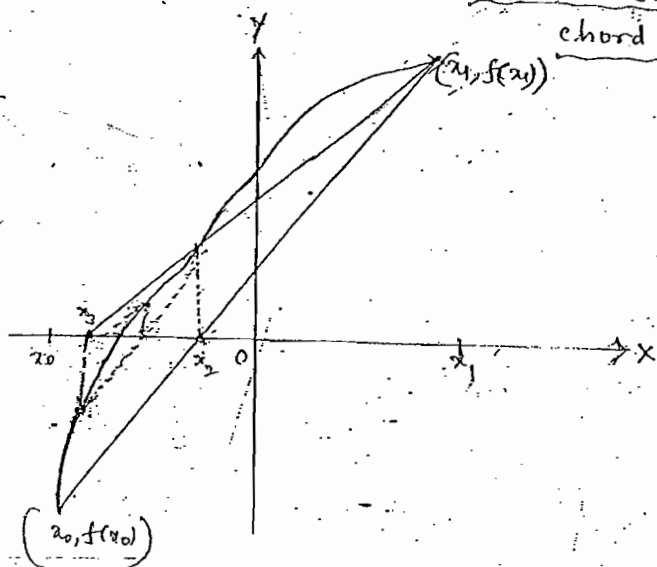
$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{(1)}$$

→ Geometrically, in Secant method, we replace the graph of $f(x)$ in the interval (x_n, x_{n+1}) by a ~~straight line joining two points~~ $(x_0, f(x_0))$ $(x_1, f(x_1))$ on the curve and take the point of intersection with x-axis as approximate value of the root.

Any line joining two points on t

is called a secant (or) line. That is why this method is known as Secant method (or) chord method

→ Determine an approximate root of the eqn $x^2 - 2x - 1 = 0$ using secant method starting with $x_0 = 2.6$ and $x_1 = 2.5$, rounded-off to 5 decimal places. Compare the result with the exact root $1+\sqrt{2}$.

Soln: Let $f(x) = x^2 - 2x - 1$

Starting with $x_0 = 2.6$ and $x_1 = 2.5$ the successive approximations are

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2.6 f(2.5) - 2.5 f(2.6)}{f(2.5) - f(2.5)}$$

$$= \frac{2.6(0.25) - (2.5)(.56)}{0.25 - 0.56} = 2.41935484$$

and $f(x_2) = 0.0145682$

To find the next approximation, we compute

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{(2.5)(0.0145682) - (2.41935484)(0.56)}{(0.0145682) - (0.56)}$$

$$= 2.41436464.$$

proceeding similarly, we get

$x_4 = 2.41421384$ and $x_5 = 2.41421356$.

Since x_4 and x_5 rounded-off to 5 decimal places are the same, we stop the process here.

∴ The required root rounded-off to 5 decimal places is 2.41421.

The exact value of the root $1+\sqrt{2} = 2.41421$, which is rounded-off to 5 decimal places. Hence the computed root and exact root are the same when we round off to five decimal places.

H.W → Determine an approximate root of the eqn $\cos x - xe^x = 0$ using secant method, starting with the two initial approximations as correct to 4 decimal places.

→ Find an approximate root of the e... $x^3 + x^2 - 3x - 3 = 0$ using
(a) regula-falsi method correct to ...
(ii) secant method starting with $x_0 =$ ... off to 3 decimal places.

b) Compare the results obtained by (i) & (ii) in part (a).

Sol: Let $f(x) = x^3 + x^2 - 3x - 3$

Take $x_0 = 1$ and $x_1 = 2$.

$f(x_0) = -4 < 0$ and $f(x_1) = 3 > 0$.

$\therefore$ The root lies between 1 and 2.

By the method of false-position, the first approximation is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{—①}$$

$$= \frac{1(3) - 2(-4)}{3 - (-4)} = \frac{11}{7} = 1.57142$$

Now $f(x_2) = -1.36449 < 0$ and $f(x_1) f(x_2) < 0$

$\therefore$ The root lies between 1.57142 and 2

Take $x_0 = 1.57142$ and $x_1 = 2$.

$f(x_0) = -1.36449$ and $f(x_1) = 3$

$\therefore$ from ①

$$x_3 = \frac{(1.57142)(3) - 2(-1.36449)}{1.57142 + 1.36449}$$

$x_3 = 1.70540.$

Now repeating this process, the successive approximations is given by

$f(x_3) = -ve < 0$, the root lies b/w 1.70540833, 2, $x_4 = 1.72788$

$x_4 = 1.72788$, $x_5 = 1.73140$, and $x_6 = 1.73194$.

Since x_5 and x_6 are correct to 3 decimal places are same.

$\therefore$ we stop the process here.

Hence the root correct to 3 decimal places is 1.731.

(ii) Secant method.

Starting with $x_0 = 1$, $x_1 = 2$ the successive approximations are

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(3) - 2(-4)}{3-(-4)} = \frac{11}{7} = 1.57142$$

To calculate the next approximation, take $x_1 = 2$ and $x_2 = 1.57142$, we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{(1.57142)(3) - 2(-1.36449)}{1.57142 + 1.36449} = 1.70540$$

To find the 3rd approximation let $x_2 = 1.57142$ and $x_3 = 1.70540$

$$x_4 = \frac{(1.57142) f(1.70540) - (1.70540) f(1.57142)}{f(1.70540) - f(1.57142)}$$

$$= \frac{(1.57142)(-0.24784) - (1.70540)(-1.36449)}{-0.24778 + 1.36449}$$

$$= 1.73573$$

Repeating this process,

we get $x_5 = 1.73199$, $x_6 = 1.73205$

Since x_5 and x_6 rounded-off to 3 decimal places are the same, we stop here

Hence the root is 1.732, rounded-off 3 decimal places.

(b) W.K.T $|x_{i+1} - x_i|$ gives the error after the ith iteration.

In Regula-falsi method, the error after 5th iteration is

$$|x_6 - x_5| = |1.73194 - 1.73140|$$
$$= 0.00011.$$

Whereas in secant method, the error after 5th iteration is

$$|x_6 - x_5| = |1.73205 - 1.73199|$$
$$= 0.00006$$

This shows that the error in the case of secant method is smaller than that in regula-falsi method for the same number of iterations.

Newton - Raphson Method :

→ This method is one of the most useful method for finding roots of an algebraic equation.

→ Suppose we want to find an approximate root of the eqn $f(x) = 0$.

- If $f(x)$ is continuous, then we can apply either bisection method or regula-falsi method to find approximate roots.

Now if $f(x)$ and $f'(x)$ are continuous, then we can use a new iteration method called Newton-Raphson method. This method gives the result more faster than bisection or regula-falsi methods.

- The underlying idea of the method is due to mathematician Isaac Newton. But the method as now used is due to the mathematician Raphson.

→ Suppose we want to find a root of the equation $f(x) = 0$ where $f(x)$ and $f'(x)$ are continuous.

Let x_0 be an initial approximation and assume that x_0 is close to the exact root α and $f'(x_0) \neq 0$.

Let $\alpha = x_0 + h$ where h is a small quantity.

Hence $f(\alpha) = f(x_0 + h) = 0$

Now, expanding $f(x_0+h)$ by Taylor's theorem, we get

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since h is small, neglecting the terms containing h^2 and higher powers, we get

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow h = \frac{-f(x_0)}{f'(x_0)}$$

This gives a new approximation to α as

$$x_1 = x_0 + h \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now the iteration can be defined by —

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

- - - - -

- - - - -

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{--- (1)}$$

which is the Newton Raphson formula.

Geometrical Interpretation

Suppose the graph of the function $y = f(x)$ crosses the x-axis at α, then $x = \alpha$ is the root of the eqn $f(x) = 0$

If x_0 is an initial approximation to the root α, then the corresponding point on the graph is $P(x_0, f(x_0))$. We draw a tangent to the curve at P, it intersect the x-axis at T. Let x_1 be the co-ordinate of T.

Let $S(\alpha, 0)$ denote the point on the x-axis where the curve cuts the x-axis, where α is the root of the eqn $f(\alpha) = 0$.

Take x_1 as the new approximation which may be closer to α than x_0.

Now let us find the tangent at $P(x_0, f(x_0))$.

The slope of the tangent at P is given by $f'(x_0)$.

$\therefore$ By the point-slope form of the expression for a tangent to a curve is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The tangent passes through the point $T(x_1, 0)$

$$\therefore 0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This x_1 is the first iterated value.

To get the second iterated value we again consider a tangent at the point $P(x_1, f(x_1))$ on the curve and repeat the process. Then we get $T(x_2, 0)$ on the x-axis.

From the figure, we observe that T_1 is more closer to $S(\alpha, 0)$ than T. Therefore after each iteration the approximation is coming closer and closer to the actual root.

5(c) Ex-1 Find a real root of the eqn $x^3 - 4x + 1 = 0$ using Newton-Raphson method, starting with $x_0 = 0$ rounded-off to 4 decimal places.

Sol^n: Let $f(x) = x^3 - 4x + 1$.

$f'(x) = 3x^2 - 4$.

Clearly $f(x)$ and $f'(x)$ are continuous everywhere.

The initial approximation is $x_0 = 0$

The Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \quad \text{---} (1)$$

putting $n = 0$, in (1) the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{1}{(-4)} = \frac{1}{4} = 0.25$$

putting $n=1$ in (1),

the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.25 - \frac{f(0.25)}{f'(0.25)}$$

$$= 0.25 - \frac{0.015625}{(-3.8125)}$$

$$= 0.254098.$$

Similarly, we get

$$x_3 = 0.254101.$$

Since x_2 and x_3 rounded-off to four decimal places are the same, we stop the iteration here.

Hence the root is 0.2541

Q→ Using Newton-Raphson method, find the real root of the eqn $x^3-6x+4=0$ lying between 0 and 1 correct to 4 decimal places.

Solⁿ: We have $f(x) = x^3-6x+4.$

$$f'(x) = 3x^2-6.$$

Clearly $f(x)$ and $f'(x)$ are continuous [0,1]

we have $f(0) = 4$ and $f(1) = -1$

$\therefore f(0)f(1) < 0$

$\therefore$ The root ~~lies b/w~~ 0 & 1.

~~The value of the root is nearer to 1.~~

Let $x_0 = 0.7$ be the approximation to the root

Now $f(x_0) = f(0.7) = 0.143$

and $f'(x_0) = f'(0.7) = -4.53$

Then by newton's iteration formula, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.7 - \frac{0.143}{(4.53)}$$

$$= 0.7316$$

Now $f(x_1) = f(0.7316) = 0.0019805$

and $f'(x_1) = f'(0.7316) = -4.39428$

∴ The second approximation of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7316 + \frac{0.0019805}{4.39428}$$

$$= 0.73250699.$$

→ Find the smallest positive root of $2x - \tan x = 0$ by Newton Raphson method, correct to 5 decimal places. [Ans: 1.16556] let $x_0 = 1$

→ By using the Newton Raphson method, find an approximate root of $2x - 2 - \sin x = 0$ by Newton Raphson method in the interval $[0, \pi]$ with error less than 10^{-5} start with $x_0 = 1.5$. [Ans: 1.498701]

→ Find a real root of the eqn $x^3 - x - 1 = 0$ using Newton-Raphson method, correct to four decimal places. [Hint: $f(1) < 0$ & $f(2) > 5$] Ans: 1.3247.

→ Find the real root of the eqn $3x = \cos x + 1$ by using Newton's-Raphson method [Ans: 0.6071] (root lies b/w 0 & 1)

→ Find the real root of the eqn $x \log_{10} x = 1.2$ correct to five decimal places. [Ans: 2.74065] [root lies b/w 2 & 3]

→ Apply Newton-Raphson's method to determine a root of the eqn $f(x) = \cos x - xe^x = 0$ such that $|f(x^*)| < 10^{-8}$, where x^* is the approximation to the root.

[Ans: $x^* = 0.51775736$] Let $x_0 = 1$.

Here $f(x^*) = -0.2910 \times 10^{-10}$

$\therefore |f(x^*)| < 10^{-8}$.

→ We shall now consider an application of Newton-Raphson formula.

W.K.T finding the square ~~root of~~ a number is not easy unless we use a calculator. Calculators use some algorithm to obtain this value.

We shall now illustrate how Newton-Raphson method enables us to obtain such an algorithm for calculating square roots.

Ex: Find an approximate value of $\sqrt{2}$ using the Newton-Raphson formula.

Sol: Let $x = \sqrt{2}$

$\Rightarrow x^2 - 2 = 0$.

Let $f(x) = x^2 - 2$.

then $f'(x) = 2x$.

Clearly $f(x)$ and $f'(x)$ are continuous everywhere.

$\therefore$ $f(x)$ satisfies all the conditions for Newton Raphson method.

$\therefore$ Choose $x_0 = 1$ be the initial approximation to the root. $\left[\because \sqrt{1} < \sqrt{2} < \sqrt{4}\right.$ $1 < \sqrt{2} < 2$) The root is nearest to 1 $]$

The iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 2}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{2}{x_n}\right] \quad \text{--- (1)}$$

putting $n = 0, 1, 2, 3, \ldots$, we get

$$x_1 = \frac{1}{2}\left[x_0 + \frac{2}{x_0}\right]$$

$$\Rightarrow x_1 = \frac{1}{2}\left[1 + \frac{2}{1}\right]$$

$$= \frac{3}{2} = 1.5$$

$$x_2 = \frac{1}{2}\left[1.5 + \frac{2}{1.5}\right] = 1.4166667$$

$$x_3 = \frac{1}{2}\left[1.4166667 + \frac{2}{1.4166667}\right]$$

$$= 1.4142157$$

Similarly, we get

$$x_4 = 1.4142136$$

$$x_5 = 1.4142136.$$

Thus the value of $\sqrt{2}$ correct to seven decimal places is 1.4142136.

Note [1] The method used in the above example is applicable for finding

square root of any +ve real number.

For example. we want to find an approximate value of $\sqrt{N}$ where N is a positive real number. Then we consider eqn $x^2 - N = 0$

The iterated formula is

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

[2]. From the above example and examples (1)&(2) we find that Newton-Raphson method gives the root very fast.

One reason for this is that the derivative $|f'(x)|$ is large compared to $|f(x)|$ for any $x = x_n$. The quantity $\left|\frac{f(x)}{f'(x)}\right|$ which is the difference between two iterated values is small in this case.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow \frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}$

→ In general we can say that if $|f'(x_i)|$ is large compared to $|f(x_i)|$, then we can obtain the desired root very fast by this method.

→ The Newton-Raphson method has some limitations. Some of the difficulties are as given below.

[1] Suppose $f'(x_i)$ is zero in a neighbourhood of the root, then it may happen that $f'(x_n) = 0$ for some x_n. In this case we cannot apply Newton-Raphson formula, since division by zero is not allowed.

[2] Another difficulty is that it may happen that $f'(x)$ is zero only at the roots.

- This happens in either of the situations.

(i) $f(x)$ has multiple root at α i.e, a polynomial function $f(x)$ has a multiple root α of order p, then $f(x)$ can be written as

$$f(x) = (x-\alpha)^p \, h(x)$$

where $h(x)$ is a function such that $h(\alpha) \neq 0$.

— For a general function $f(x)$, this means

$$f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots\cdots = f^{p-1}(\alpha) = 0$$

and $f^p(\alpha) \neq 0$.

(ii) $f(x)$ has a stationary point (point of maximum or minimum) at the root.

i.e, $f'(x) = 0$ at some point $x = x_n$.

H-W → Using Newton-Raphson method find the

(i) square root of 8. **Ans: 2.828427**

(ii) square root of $\sqrt{28}$ **Ans: 5.2915**

→ Using Newton Raphson method prove that

1997 (i) Iterative formula for $\frac{1}{N}$ is $x_{n+1} = x_n(2 - Nx_n)$

(ii) Iterative formula for $\frac{1}{\sqrt{N}}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$

(iii) Iterative formula for $\sqrt[K]{N}$ is $x_{n+1} = \frac{1}{K}\left[(K-1)x_n + \frac{N}{x_n^{K-1}}\right]$

Sol^n: (i) Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x}$

$$\Rightarrow \frac{1}{x} - N = 0$$

Let $f(x) = \frac{1}{x} - N$.

then $f'(x) = -\frac{1}{x^2} = -x^{-2}$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\left(-x_n^{-2}\right)}$$

$$= x_n + \left(\frac{1}{x_n} - N\right) x_n^2$$

$$= x_n + x_n - N x_n^2$$

$$= 2x_n - N x_n^2$$

$$\boxed{x_{n+1} = x_n(2 - N x_n)}.$$

which is the required result.

(ii) Let $x = \frac{1}{\sqrt{N}}$

$\Rightarrow x^2 = \frac{1}{N}$

$\Rightarrow x^2 - \frac{1}{N} = 0$

Let $f(x) = x^2 - \frac{1}{N}$

then $f'(x) = 2x$.

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(x_n^2 - \frac{1}{N}\right)}{2x_n}$$

$$= \frac{\left(2x_n^2 - x_n^2 + \frac{1}{N}\right)}{2x_n}$$

$$= \frac{x_n^2 + \frac{1}{N}}{2x_n} = \frac{1}{2}\left[x_n + \frac{1}{N x_n}\right]$$

(iii) Let $x = \sqrt[k]{N} \Rightarrow x^k = N$

$\Rightarrow x^k - N = 0$

Let $f(x) = x^k - N$.

Then $f'(x) = Kx^{K-1}$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^K - N}{Kx_n^{K-1}}$$

$$= \frac{Kx_n^K - x_n^K + N}{Kx_n^{K-1}} = \frac{1}{K}\left[(K-1)x_n^K + \frac{N}{x_n^{K-1}}\right]$$

→ Evaluate the following (correct to four decimal places) by Newton-Raphson method.

(i) $1/31$ (ii) $\frac{1}{\sqrt{14}}$ (iii) $\sqrt[3]{24}$ (iv) $(30)^{-1/5}$ [Hint: put $K=-5$ in formula(iii)]

[Ans: 0.0323] [Ans: 0.2673] [Ans: 2.8845] [Ans: 0.5065]

Using Newton-Raphson's method, show that the iteration formula for finding the reciprocal of the p^{th} root of N is

$$x_{n+1} = \frac{x_n(p+1 - Nx_n^p)}{p}.$$

Soln: $x = \frac{1}{\sqrt[p]{N}} \Rightarrow x = \frac{1}{N^{1/p}} \Rightarrow x = N^{-1/p} \Rightarrow x^{-p} = N$
$\Rightarrow x^{-p} - N = 0.$

Let $f(x) = x^{-p} - N$

$f'(x) = -p\,x^{-p-1}$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^{-p} - N}{-p\,x_n^{-p-1}}$$

$$= \frac{px_n + (x_n^{-p} - N)x_n^{p+1}}{p}$$

$$= \frac{px_n + x_n - Nx_n^{p+1}}{p} = \frac{x_n(p+1 - Nx_n^p)}{p}$$

which is the required result.

Convergence criterion

We shall now introduce a new concept called convergence criterion related to an iteration process. This criterion gives us an idea of ~~how many successive~~ how many successive iterations have to be carried out to obtain the desired accuracy.

Definition: Let $x_0, x_1, \ldots, x_n, \ldots$ be the successive approximations of an iteration process; we denote the sequence of these approximations as $\{x_n\}_{n=0}^{\infty}$. We say that $\{x_n\}_{n=0}^{\infty}$ converges to a root α with order $p \geq 1$ if

$$|x_{n+1} - \alpha| \leq \lambda |x_n - \alpha|^p \quad \text{--- (1)}$$

for some number $\lambda > 0$. p is called the order of convergence and λ is called the ~~asyptotic~~ asymptotic error constant.

For each n, we denote by $\epsilon_n = x_n - \alpha$. Then the eqn (1) be written as

$$|\epsilon_{n+1}| \leq \lambda |\epsilon_n|^p \quad \text{--- (2)}$$

This inequality shows the relationship between the error in successive approximations.

For example:

- Suppose $p=2$ and $|\epsilon_n| \simeq 10^{-2}$ for so[illegible]

we can expect that $|\epsilon_{n+1}| \simeq \lambda 1$[illegible]

Thus if p is large, the iterat[illegible]

rapidly.

When p takes the values 1, 2, 3 then we say that the convergence is linear, quadratic and cubic respectively.

In the case of linear convergence (i.e. $p=1$), then we require that $\lambda < 1$.

$\therefore$ Eqn (1) becomes

$$|x_{n+1}-\alpha| \leq \lambda |x_n - \alpha| \text{ for all } n \geq 0 \quad \text{--- (3)}$$

If this condition is satisfied for an iteration process then we say that the iteration process converges linearly.

Setting $n=0$, in the inequality (3), we get

$$|x_1 - \alpha| \leq \lambda |x_0 - \alpha|.$$

For $n=1$, we get

$$|x_2-\alpha| \leq \lambda |x_1 - \alpha|$$
$$\leq \lambda^2 |x_0 - \alpha|$$

For $n=2$

$$|x_3-\alpha| \leq \lambda |x_2-\alpha|$$
$$\leq \lambda^2 |x_1 - \alpha|$$
$$\leq \lambda^3 |x_0 - \alpha|.$$

Using induction on 'n', we get

$$|x_n-\alpha| \leq \lambda^n |x_0-\alpha| \quad \text{--- (4)}$$

If either of the inequalities (3) or (4) is satisfied, then we conclude that $\{x_n\}_{n=0}^{\infty}$ cgs to the root.

Convergence of bisection method:

Suppose that we apply the bisection method on the interval $[a_0, b_0]$. For the eqn $f(x)=0$. In this method we construct intervals $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots\ldots$ each of which contains the required root of the given eqn. In each step the interval width is reduced by $\frac{1}{2}$.

i.e.

$$b_1 - a_1 = \frac{b_0 - a_0}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}$$

$$\text{-----}$$

$$b_n - a_n = \frac{b_0 - a_0}{2^n} \quad \text{———— (5)}$$

Clearly the eqn $f(x)=0$ has a root in $[a_0, b_0]$. Let α be the root of the eqn. Then α lies in ~~all the~~ in all the intervals $[a_i, b_i]$, $i = 0, 1, 2, \ldots$

for any n, let $c_n = \frac{a_n + b_n}{2}$ denote the middle point of the interval $[a_n, b_n]$. Then $c_0, c_1, c_2, \ldots$ are taken as successive approximations to the root α.

Let us check the inequality (3) for $\{c_n\}_{n\geq 0}$.

for each n, α lies in the interval $[a_n, b_n]$

$\therefore$ we have

$$|c_{n+1} - \alpha| \leq \frac{|c_n - \alpha|}{2}$$

Thus $\{c_n\}_{n\geq 0}^{\infty}$ cgs to the root α. Hence

Say that the bisection method always cgs.

→ for practical purposes, we should be able to decide at what stage we can stop the iteration to have an acceptably good approximate value of α. The number of iterations required to achieve a given accuracy for the bisection method can be obtained.

→ Suppose that we want an approximate solution with an error bound of 10^{-M}.

Taking logarithms on both sides of eqn ⑤ we find the number of iterations required, say n, approximately given by

$$n = \text{int}\left[\frac{\log(b_0 - a_0) - \log 10^{-M}}{\log 2}\right] \quad \text{——— ⑥}$$

$\frac{b_0 - a_0}{2^n} \leq 10^{-M}$

$\log\left[\frac{b_0 - a_0}{2^n}\right] = \log 10^{-M}$

where the symbol 'int' stands for the integral part of the number in the bracket and $[a_0, b_0]$ is the initial interval in which a root lies.

since n is integer, we take n as the next nearest integer

Ex 1 Suppose that the bisection method is used to find a zero of $f(x)$ in the interval $[0,1]$. How many times this interval be bisected to guarantee that we have an approximate root with absolute error less than or equal to 10^{-5}.

Sol: ~~Let 'n' denote the~~ required number.

$a_0 = 0$, $b_0 = 1$ and $M = 5$.

from eqn ⑥

$$n = \text{int}\left[\frac{\log(b_0 - a_0) - \log 10^{-M}}{\log 2}\right]$$

$$n = \text{int}\left[\frac{\log 1 - \log 10^{-5}}{\log 2}\right]$$

$$= \text{int}\left[\frac{11.51292542}{0.69314718}\right]$$

$$= \text{int}[16.60964047]$$

$n = 17$ (approximately).

→ The following table gives the minimum no. of iterations required to find an approximate root in the interval [0,1] for various acceptable errors.

E	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
n	7	10	14	17	20	24

This table shows that for getting an approximate value with an absolute error bounded by 10^{-5}, we have to perform 17 iterations.

– Thus even though the bisection method is simple to use, it requires a large no. of iterations to obtain a reasonably good approximate root. This is one of the disadvantages of the bisection method.

Convergence criteria for Secant Method:

Let $f(x)=0$ be the given eqn. Let α denote a simple root of the eqn $f(x)=0$. Then we have $f'(\alpha)\neq 0$.

The iteration formula for the Secant method is

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} f(x_i) \quad \text{——— (i)}$$

For each i, set $\epsilon_i = x_i - \alpha$.

$\Rightarrow x_i = \epsilon_i + \alpha$.

Substituting $x_i = \epsilon_i + \alpha$ in eqn (i)

$$\epsilon_{i+1} + \alpha = \epsilon_i + \alpha - \frac{\epsilon_i - \epsilon_{i-1}}{f(\epsilon_i+\alpha) - f(\epsilon_{i-1}+\alpha)} f(\epsilon_i+\alpha)$$

$$\epsilon_{i+1} = \epsilon_i - \frac{\epsilon_i - \epsilon_{i-1}}{f(\epsilon_i+\alpha) - f(\epsilon_{i-1}+\alpha)} f(\epsilon_i+\alpha) \quad \text{——— (ii)}$$

Now expanding $f(\epsilon_i+\alpha)$ and $f(\epsilon_{i-1}+\alpha)$ using Taylor's theorem about the point $x=\alpha$

we get

$$f(\epsilon_i+\alpha) = f(\alpha) + \frac{f'(\alpha)}{1}\epsilon_i + \frac{f''(\alpha)}{2}\epsilon_i^2 + \cdots$$

$$= f'(\alpha)\epsilon_i + \frac{f''(\alpha)}{2}\epsilon_i^2 + \cdots \quad (\because f(\alpha)=0)$$

$$= f'(\alpha)\left[\epsilon_i + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_i^2 + \cdots\right] \quad \text{——— (iii)}$$

Similarly

$$f(\epsilon_{i-1}+\alpha) = f'(\alpha)\left[\epsilon_{i-1} + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_{i-1}^2 + \cdots\right] \quad \text{——— (iv)}$$

$$\therefore f(\epsilon_i+\alpha) - f(\epsilon_{i-1}+\alpha) = f'(\alpha)\left[(\epsilon_i - \epsilon_{i-1}) + (\epsilon_i^2 - \epsilon_{i-1}^2)\frac{f''(\alpha)}{2f'(\alpha)} + \cdots\right]$$

$$= f'(\alpha)(\epsilon_i - \epsilon_{i-1})\left[1 + (\epsilon_i + \epsilon_{i-1})\frac{f''(\alpha)}{2f'(\alpha)} + \cdots\right] \quad \text{——— (v)}$$

Substituting eqns (iii) & (v) in eqn (ii), we get

$$\epsilon_{i+1} = \epsilon_i - \frac{(\epsilon_i - \epsilon_{i-1}) f'(\alpha)\left[\epsilon_i + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_i^2 + \cdots\right]}{f'(\alpha)(\epsilon_i - \epsilon_{i-1})\left[1 + (\epsilon_i + \epsilon_{i-1})\frac{f''(\alpha)}{2f'(\alpha)} + \cdots\right]}$$

$$= \epsilon_i - \left[\epsilon_i + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_i^2 + \cdots\right]\left[1 + \frac{1}{2}(\epsilon_i + \epsilon_{i-1})\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$= \epsilon_i - \left[\epsilon_i + \frac{1}{2}\epsilon_i^2\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 - \frac{1}{2}(\epsilon_i + \epsilon_{i-1})\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$= \epsilon_i - \left[\epsilon_i + \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}(\epsilon_i^2 - \epsilon_i^2 - \epsilon_i\epsilon_{i-1}) + \cdots\right]$$

By neglecting the terms involving $\epsilon_i\epsilon_{i-1}^2 + \epsilon_i^2\epsilon_{i-1}$ in the above expression, we get

$$\epsilon_{i+1} \approx \epsilon_i\epsilon_{i-1}\left[\frac{f''(\alpha)}{2f'(\alpha)}\right] \quad \text{——— (vi)}$$

This relationship between the errors is called error eqn. ~~This relationship~~ holds only if α is a simple root.

Now using eqn (vi) we will find the numbers p and λ such that

$$\epsilon_{i+1} = \lambda\epsilon_i^p; \quad i = 0, 1, 2, \cdots \quad \text{——— (vii)}$$

Setting $i = j-1$, we obtain

$$\epsilon_j = \lambda\epsilon_{j-1}^p$$

(or) $$\epsilon_i = \lambda\epsilon_{i-1}^p \quad \text{——— (viii)}$$

Taking p^{th} root on both sides of (viii), we get

$$\epsilon_i^{1/p} = \lambda^{1/p}\epsilon_{i-1}$$

$$\Rightarrow \epsilon_{i-1} = \lambda^{-1/p}\epsilon_i^{1/p} \quad \text{——— (ix)}$$

from eqns (vi) & (vii) ; we have

$$\lambda \epsilon_i^p = \epsilon_i \epsilon_{i-1} \frac{f''(\alpha)}{2f'(\alpha)}$$

$$\Rightarrow \lambda \epsilon_i^p = \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_i \lambda^{-1/p} \epsilon_i^{1/p} \quad (\text{by eqn (ix)})$$

$$\Rightarrow \lambda \epsilon_i^p = \frac{f''(\alpha)}{2f'(\alpha)} \lambda^{-1/p} \epsilon_i^{1+\frac{1}{p}} \text{ ——— (x)}$$

equating the powers of ϵ_i on both sides of eqn (x)

we get $p = 1 + \frac{1}{p}$

$\Rightarrow p^2 - p - 1 = 0$

which gives $p = \frac{1 \pm \sqrt{5}}{2}$ ($\because p$ cannot be -ve), neglecting the minus sign

$$\boxed{p = \frac{1+\sqrt{5}}{2} \simeq 1.618}$$

Now, to get the number λ, we equate the constant terms on both sides of eqn (x),

we get $\lambda = \frac{f''(\alpha)}{2f'(\alpha)} \lambda^{-1/p}$

$$\Rightarrow \lambda^{1+\frac{1}{p}} = \frac{f''(\alpha)}{2f'(\alpha)}$$

$$\Rightarrow \boxed{\lambda = \left[\frac{f''(\alpha)}{2f'(\alpha)}\right]^{p/p+1}}$$

Hence the order of convergence of the secant method is $p = 1.62$ and the asymptotic error constant is $\left[\frac{f''(\alpha)}{2f'(\alpha)}\right]^{p/p+1}$

Ex: The following are the five successive iterations obtained by secant method to find the root $\alpha = 2$ of the eqn $x^2 - 3x + 2 = 0$.

$x_1 = -2.6$, $x_2 = -2.4$, $x_3 = -2.106598985$
$x_4 = -2.022641412$ and $x_5 = -2.000022537$

Compute the asymptotic error constant and show that $\epsilon_5 \simeq \lambda \epsilon_4$.

Sol: Let $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

$\therefore f'(-2) = 9$ and $f''(-2) = -12$

we have $\lambda = \left[\frac{f''(\alpha)}{2f'(\alpha)}\right]^{P/1+P}$

$\lambda = \left[\frac{-12}{18}\right]^{\frac{1.62}{1+1.62}} = \left(\frac{-2}{3}\right)^{\frac{1.62}{2.62}}$

$= \left(\frac{-2}{3}\right)^{0.618}$

$\lambda = -0.778351205$

Now $\epsilon_5 = |x_5 - \alpha|$

$= |-2.000022537 + 2|$

$= 0.000022537$

and $\epsilon_4 = |x_4 - \alpha|$

$= |-2.022641412 + 2|$

$= 0.022641412$

Then $\lambda \epsilon_4 = 0.778351205 \times 0.022641412$

$= 0.000021246$

≈ 0.00002253

$\approx \epsilon_5$

$\lambda \epsilon_4 \simeq \epsilon_5$

Convergence of Newton-Raphson Method :

Newton-Raphson iteration formula is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{---} (1)$$

To obtain the order of convergence, assume that α is a simple root of $f(\alpha)=0$.

Let $x_i - \alpha = \epsilon_i$, $i = 0, 1, 2, \ldots$

Also $x_{i+1} - \alpha = \epsilon_{i+1}$

$\therefore$ from (1)

$$\epsilon_{i+1} + \alpha = \epsilon_i + \alpha - \frac{f(\epsilon_i + \alpha)}{f'(\epsilon_i + \alpha)}$$

$$\Rightarrow \epsilon_{i+1} = \epsilon_i - \frac{f(\epsilon_i+\alpha)}{f'(\epsilon_i+\alpha)} = \frac{\epsilon_i f'(\epsilon_i+\alpha) - f(\epsilon_i+\alpha)}{f'(\epsilon_i+\alpha)}$$

Now expanding $f(\epsilon_i+\alpha)$ and $f'(\epsilon_i+\alpha)$, using Taylor's theorem, about the point α, we obtain

$$\epsilon_{i+1} = \frac{\epsilon_i\left[f'(\alpha) + \epsilon_i f''(\alpha) + \frac{\epsilon_i^2}{2} f'''(\alpha) + \cdots\right] - \left[f(\alpha) + \epsilon_i f'(\alpha) + \frac{\epsilon_i^2}{2} f''(\alpha) + \cdots\right]}{f'(\alpha) + \epsilon_i f''(\alpha) + \epsilon_i^2 f'''(\alpha) + \cdots}$$

But $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

$$\therefore \epsilon_{i+1} = \frac{\frac{\epsilon_i^2}{2} f''(\alpha) + \cdots}{f'(\alpha)\left[1 + \epsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]}$$

$$= \left[\frac{\epsilon_i^2}{2} f''(\alpha) + \cdots\right] \frac{1}{f'(\alpha)} \left[1 + \epsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$= \frac{1}{f'(\alpha)} \left[\frac{\epsilon_i^2}{2} f''(\alpha) + \cdots\right] \left[1 - \epsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

On neglecting ϵ_i^3 and higher power of ϵ_i,

we get $\epsilon_{i+1} \simeq \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_i^2$

This shows that the errors satisfy the inequality $|\epsilon_{i+1}| \le \lambda |\epsilon_i|^p$ with;

$$p = 2 \text{ and } \lambda = \frac{f''(\alpha)}{2f'(\alpha)}$$

Hence Newton Raphson method is of order 2

i.e, the Newton Raphson method has second order convergence.

and the error is proportional to the square of the previous error in each step.

Note: If α is a multiple root i.e, $f'(\alpha)=0$, then the convergence is not quadratic, but only linear.

For example: Let $f(x) = (x-2)^4 = 0$ starting with the initial approximation $x_0 = 2.1$, compute the iterations x_1, x_2, x_3 and x_4 using Newton-Raphson method. Is the sequence converging quadratically or linearly?

Soln: Let $f(x) = (x-2)^4$. The given function has multiple roots at $x=2$ is of order 4.

Newton Raphson iteration formula for the given equation is

$$x_{i+1} = x_i - \frac{(x_i-2)^4}{4(x_i-2)^3}$$

$$= x_i - \frac{1}{4}(x_i - 2) = \frac{1}{4}(3x_i + 2)$$

Starting with $x_0 = 2.1$, the iterations are given by

$$x_1 = \frac{1}{4}(6.3+2) = \frac{8.3}{4} = 2.075$$

Similarly, $x_2 = 2.05625$

$x_3 = 2.0421875$

$x_4 = 2.031640625$

Now $\epsilon_0 = x_0 - 2 = 0.1$

$\epsilon_1 = x_1 - 2 = 0.075$

$\epsilon_2 = x_2 - 2 = 0.05625$

$\epsilon_3 = x_3 - 2 = 0.0421875$

$\epsilon_4 = x_4 - 2 = 0.031640625$

where $\epsilon_1 = 0.075$

$= \frac{3}{4} \times 0.1$

$= \frac{3}{4}\epsilon_0$

$\therefore \epsilon_1 = \frac{3}{4}\epsilon_0$

and $\epsilon_2 = \frac{3}{4}\epsilon_1$

$\epsilon_3 = \frac{3}{4}\epsilon_2$

$\epsilon_4 = \frac{3}{4}\epsilon_3$

$e_{n+1} = \lambda(e_n)^p$ type.

i.e, the convergence is linear in this case.

Also the error is reduced by a factor of $\frac{3}{4}$ with each iteration.

H.W. → The quadratic eqn $x^4 - 4x^2 + 4 = 0$ has a double root at $x = \sqrt{2}$. Starting with $x_0 = 1.5$, compute three successive iterations to the root by Newton-Raphson method. Does the result converge quadratically or linearly?

Set-III

Solution of System of linear equations

Introduction:

System of linear equations arise in a large number of areas, both directly in modelling physical situations and indirectly in the numerical solution of other mathematical models.
These applications occur in all areas of the physical, biological and engineering sciences.
For instance, in physics, the problem of steady state temperature in a plate is reduced to solving linear equations.
Linear algebraic systems are also involved in the optimization theory, least squares fitting of data, numerical solution of boundary value problems for ordinary and partial differential eqns, statistical inference etc. The numerical solution of systems of linear algebraic eqns play a very important ...

Numerical methods for solving linear algebraic systems may be divided into two types, direct and iterative.
Direct methods are those which, in the absence of round-off or other errors, yield the exact solution in a finite number of elementary operations.
Iterative methods start with an initial approximation and by applying a suitably chosen ... lead to successively better approx.

The general form of a system of 'm' linear eqns in 'n' unknowns $x_1, x_2, \ldots, x_n$ can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \text{---} (1)$$

$$\Rightarrow AX = B \quad \text{---} (2)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1}$

- The solution of the system of eqns (2) gives 'n' unknown values $x_1, x_2, \ldots, x_n$ which satisfy the system simultaneously.
- A system of eqns (2) is said to be consistent if it has atleast ~~one~~ solution. ~~If no solution~~ exists, then the system is said to be inconsistent.
- The system of eqns (2) is said to be homogeneous if $b = 0$, that is all the elements $b_1, b_2, \ldots b_n$ are zero, otherwise the system is called non-homogeneous.
- In this lesson, we consider only non-homogeneous, and we restrict $m = n$ (i.e, the number of eqns = the no. of unknowns)
- A non-homogeneous system of n linear eqns in 'n' unknowns has a unique solution iff the coefficient matrix A is non-singular. (ie, $|A| \neq 0$)

If A is non-singular, then A^{-1} exists and the solution of system (2) can be expressed as $x = A^{-1}b$.
In case the matrix A is singular, then the system (2) has no solution if $b \neq 0$ or has an infinite number of solutions if $b = 0$.
Here, we assume that A is non-singular matrix.

The methods of solution of the system (2) may be classified into two types:

(i) Direct Methods: which in the absence of round-off errors give the exact solution in a finite number of steps.

(ii) Iterative Methods: Starting with an approximate solution vector $x^{(0)}$, these methods generate a sequence of approximate solution vectors $\{x^{(k)}\}$ which converge to the exact solution vector x as the number of iterations $k \to \infty$.

Thus iterative methods are infinite processes. Since we perform only a finite number of iterations, these methods can only find some approximation to the solution vector x.

Direct Methods for Special Matrices:

We now discuss three special forms of matrix A in eqn (2) for which the solution vector x can be obtained directly.

Case (i): $A = D$, where D is a diagonal matrix.
In this case the system of eqns (2) are the form

$$\left.\begin{array}{l} a_{11}x_1 \cdots\cdots = b_1 \\ \quad a_{22}x_2 \cdots = b_2 \\ \quad \ddots \\ \quad\quad a_{nn}x_n = b_2 \end{array}\right\} \quad (3)$$

and $|A| = \det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdots\cdots a_{nn}$

Since the matrix A is non-singular, $a_{ii} \neq 0$ for $i = 1, 2, 3, \cdots n$ and we obtain the solution as $x_i = \frac{b_i}{a_{ii}}$, $i = 1, 2, 3, \cdots\cdots n$.

Case(ii): $A = L$, where L is a lower triangular matrix ($a_{ij} = 0$, $j > i$). The system of eqns (2) is now of the form

$$\left.\begin{array}{l} a_{11}x_1 \qquad\qquad\qquad = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \quad (4)$$

and $|A| = a_{11}a_{22} \cdots\cdots a_{nn}$

Since the coefficient matrix A is non-singular, $a_{ii} \neq 0$, $i = 1, 2, \cdots n$.

Solving the first eqn and then successively solving second, third and so on, we obtain

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_2 = \frac{(b_2 - a_{21}x_1)}{a_{22}}$$

$$x_3 = \frac{(b_3 - a_{31}x_1 - a_{32}x_2)}{a_{33}}$$

$$\vdots$$

$$x_n = \left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j\right)\frac{1}{a_{nn}}$$

In general, we have for any i, $x_i = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}}$

$i = 1, 2, \ldots n.$

Since the unknowns in this method are solved by forward substitution, this method is called the forward substitution method.

Case (iii): $A = U$, where U is an upper triangular matrix ($a_{ij} = 0,\ j < i$). The system (2) is now of the form

$$
\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{(n-1)(n-1)}x_{n-1} + a_{(n-1)n}x_n &= b_{n-1} \\
a_{nn}x_n &= b_n
\end{aligned}\right\} \quad (5)
$$

and $|A| = a_{11}a_{22}\cdots\cdots a_{nn}$.

Since the coefficient matrix A is non-singular

$a_{ii} \neq 0$, $i = 1, 2, \ldots n$.

Solving unknowns in the order $x_n, x_{n-1}, \ldots x_2, x_1$, we get

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{(b_{n-1} - a_{(n-1)n}x_n)}{a_{(n-1)(n-1)}}$$

$$\vdots$$

$$x_1 = \left(b_1 - \sum_{j=2}^{n} a_{1j}x_j\right) \Big/ a_{11}$$

In general we have for any i, $x_i = \dfrac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{\ldots}$

The unknowns are solved by back substit... and this method is called the back substit... method.

Thus, the eqns (2) are exactly solvable, ... in (2) can be transformed into any ...

Direct Methods:-

Gaussian Elimination Method:-

In the Gaussian elimination method, the solution to the system of eqns (2) is obtained in two stages.

In the first stage, the given system of eqns is reduced to an equivalent upper triangular form using elementary transformations. In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order $x_n, x_{n-1}, x_{n-2}, \dots x_2, x_1$.

This method is explained by considering a system of 'n' eqns in 'n' unknowns in the form as follows.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = b_n \end{array}\right\} \quad \text{--- (6)}$$

Stage I: We divide the first eqn by a_{11} and then subtract this eqn multiplied by $a_{21}, a_{31}, \dots a_{n1}$ from the $2^{nd}, 3^{rd} \dots n^{th}$ eqn. Then the system (6) reduces to the following form:

$$\left.\begin{array}{l} x_1 + a'_{12}x_2 + \dots\dots + a'_{1n}x_n = b'_1 \\ \quad a'_{22}x_2 + \dots\dots + a'_{2n}x_n = b'_2 \\ \vdots \\ \quad a'_{n2}x_2 + \dots\dots + a'_{nn}x_n = b'_n \end{array}\right\} \quad \text{--- (7)}$$

Here, we can observe that the last $(n-1)$ eqns are independent of x_1, i.e, x_1 is eliminated from the last $(n-1)$ eqns.

This procedure is repeated with the second eqn of (7) i.e, we divide the second eqn by a'_{22} and then x_2 is eliminated from 3^{rd}, 4^{th}, ..., n^{th} eqns of (7). The same procedure is repeated again and again till the given system assumes the following upper triangular form:

$$\left.\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots\cdots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \cdots\cdots + c_{2n}x_n &= d_2 \\ \vdots \qquad & \quad \vdots \\ c_{nn}x_n &= d_n \end{aligned}\right\} \quad (8)$$

Stage II:

Now, the values of the unknowns are determined by back substitution procedure, in which we obtain x_n from the last eqn of (8) and then substituting this value of x_n in the preceding eqn, we get the value of x_{n-1}. Continuing this way, we can find values of all other unknowns in the order $x_n, x_{n-1}, \ldots, x_2, x_1$.

In this method, we observe that the determinant of the coefficient matrix is obtained as a by-product, i.e,

$$|A| = c_{11} \cdot c_{22} \cdots\cdots c_{nn}.$$

Example: Solve the following system of eqns using Gaussian elimination method.

$$\left.\begin{array}{l} 2x + 3y - z = 5 \\ 4x + 4y - 3z = 3 \\ -2x + 3y - z = 1 \end{array}\right\} \longrightarrow (i)$$

Soln: The given system of eqns (i) is solved in two stages.

Stage I (Reduction to upper-triangular form):-

we divide the first eqn by '2' and then subtract the resulting eqn (multiplied by 4 and −2) from the second eqn and third eqn respectively. Thus, we eliminate x from the 2nd and 3rd eqns.

The resulting new system is given by

$$\left.\begin{array}{l} x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\ -2y - z = -7 \\ 6y - 2z = 6 \end{array}\right\} \longrightarrow (ii)$$

Now, we divide the second eqn of (ii) by −2 and eliminate y from the last eqn and the modified system is given by

$$\left.\begin{array}{l} x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\ y + \frac{z}{2} = 7/2 \\ -5z = -15 \end{array}\right\} \longrightarrow (iii)$$

Stage II (Back substitution):-

from the last eqn of (iii)

we get $\boxed{z = 3}$

Using this value of z, the second eqn of (iii) gives,

$y = \frac{7}{2} - \frac{3}{2} = 2$

$\Rightarrow \boxed{y = 2}$

Using these values of y and z in the first eqn of (iii), we get -

$\boxed{x = 1}$.

Thus, the solution of the given system is $x = 1,\ y = 2,\ z = 3$.

Note: We can write the above procedure more conveniently in matrix form. Since the arithmetic operations we have performed here affect only the elements of the matrix A and the matrix B, we consider the augmented matrix i.e, $[A|B]$ and perform the elementary row operations on the augmented matrix.

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & a'_{32} & a'_{33} & b'_3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1 \\ R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a^2_{33} & b^2_3 \end{array}\right] \quad R_3 \to R_3 - \frac{a'_{32}}{a'_{22}} R_2$$

i.e, $[A|B] \xrightarrow[\text{elimination}]{\text{Gaussian}} [U|C]$

which is in desired form.

where $a'_{22}, a'_{23}, a'_{32}, a'_{33}, b'_2, b'_3, a^2_{33}, b^2_3$ are given by eqns (F) & (B).

→ The diagonal elements a_{11}, a_{22}^1 and a_{33}^2 which have been assumed to be non-zero are called pivot elements.

→ we observe that for a linear system of order 3, the elimination was performed in $3-1=2$ stages.
In general for a system of 'n' eqns given by eqns ②, the elimination is performed in $(n-1)$ stages.

At the i^{th} stage of elimination, we eliminate x_i starting from $(i+1)^{th}$ row upto the n^{th} row.
Some times, it may happen that the elimination process stops in less than $(n-1)$ stages.

But this is possible only when no eqns containing the unknowns are left or when the coefficients of all the unknowns in remaining eqns become zero. Thus if the process stops at the r^{th} stage of elimination, then we get a derived system of the form,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{22}^1 x_2 + \cdots\cdots + a_{2n}^1 x_n = b_2^1$$
$$\vdots$$
$$a_{rr}^{(r-1)} x_r + \cdots\cdots + a_{rn}^{(r-1)} x_n = b_r^{(r-1)}$$
$$0 = b_{r+1}^{(r-1)}$$
$$\vdots$$
$$0 = b_n^{(r-1)}$$

where $r \le n$ and $a_{11} \ne 0,\ a_{22}^1 \ne 0, \cdots a_{rr}^{(r-1)} \ne 0$.

In the solution of system of linear eqns we can expect two different situations.
(i) $r=n$ (ii) $r<n$.

Ex (1) Solve the following system of eqns by using Gaussian elimination method.

$$\left.\begin{aligned} 4x_1+x_2+x_3 &= 4 \\ x_1+4x_2-2x_3 &= 4 \\ -x_1+2x_2-4x_3 &= 2 \end{aligned}\right\}$$

Solⁿ: we have

$$[A|B] = \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 1 & 4 & -2 & 4 \\ -1 & 2 & -4 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 15/4 & 9/4 & 3 \\ 0 & 9/4 & 15/4 & 3 \end{array}\right] \begin{array}{l} R_2 \to R_2 - \frac{1}{4}R_1 \\ R_3 \to R_3 + \frac{R_1}{4} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & \frac{15}{4} & -\frac{9}{4} & 3 \\ 0 & 0 & -\frac{12}{5} & \frac{6}{5} \end{array}\right] \quad R_3 \to R_3 - \frac{3}{5}R_2$$

Using back substitution method, we get

$$x_3 = -\tfrac{1}{2},\ x_2 = \tfrac{1}{2},\ x_1 = 1$$

Also $|A| = -36$.

Thus in this case we observe that $r=n=3$ and the given system of eqn has a unique solution. Also the coefficient matrix is non-singular.

Ex (2): Solve the system of eqns

$$\begin{aligned} 3x_1+2x_2+x_3 &= 3 \\ 2x_1+x_2+x_3 &= 0 \\ 6x_1+2x_2+4x_3 &= 6 \end{aligned}$$

Using Gauss elimination method. Does the solution exist ?

Solⁿ: we have

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{2}{3}R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & -1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \begin{array}{l} \to b_1 \\ \to b_2' \\ \to b_3' \end{array} \quad R_3 \to R_3 - 6R_2$$

In this case $r<n$ and elements b_1, b_2' and b_3' are all non-zero. Since we cannot determine x_3 from the last eqn, the system has no solution.

In such situation we say that the eqns are inconsistent. Also $|A|=0$.

i.e, the coefficient matrix is singular.

Ex(3):

Solve the system of eqns —

$$16x_1 + 22x_2 + 4x_3 = -2$$
$$4x_1 - 3x_2 + 2x_3 = 9$$
$$12x_1 + 25x_2 + 2x_3 = -11$$

using Gauss elimination method.

Soln: we have

$$[A|B] = \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 4 & -3 & 2 & 9 \\ 12 & 25 & 2 & -11 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 0 & -\frac{17}{2} & 1 & 19/2 \\ 0 & \frac{17}{2} & -1 & -19/2 \end{array}\right] \quad R_2 \to R_2 - \frac{1}{4}R_1$$

$$\sim \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 0 & -\frac{17}{2} & 1 & \frac{19}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now in this case $r<n$ and elements b_1, b_1' are non-zero, but $b_3^{(2)}$ is zero.

Also the last eqn is satisfied for any value of x_3.

Thus we get x_3 = any value

$$x_2 = -\frac{2}{17}\left(\frac{19}{2} - x_3\right)$$

$$x_1 = \frac{1}{16}(-2 - 22x_2 - 4x_3).$$

Hence the system of eqns has infinitely many solutions.

Also $|A| = 0$.

→ We now summarise these conclusions as follows:

(i) If $r = n$ then the system of eqns (2) has unique solution which can be obtained by using the back substitution method. Moreover the coefficient matrix A in this case is non-singular.

(ii) If $r < n$ and all the elements $b_{r+1}^{(r-1)}, b_{r+2}^{(r-1)}, \ldots b_n^{(r-1)}$ are not zero then the system has no solution.

In this case we say that the system of eqns is inconsistent.

(iii) If $r < n$ and all the elements $b_{r+1}^{(r-1)}, b_{r+2}^{(r-1)}, \ldots b_n^{(r-1)}$, if present, are zero, then the system has infinite number of solutions.

In this case the system has only r linearly independent rows.

In both the cases (ii) and (iii), the matrix A is singular.

→ Use the Gaussian elimination method solve the following system of eqns

(1) $x_1 + 2x_2 + x_3 = 3$
$3x_1 - 2x_2 - 4x_3 = -2$
$2x_1 + 3x_2 - x_3 = -6$

Ans: $x_3 = 5$, $x_2 = -3$

(2) $3x_1+18x_2+9x_3=18$
$2x_1+3x_2+3x_3=117$
$4x_1+x_2+2x_3=283$

Ans: $x_3=4,\ x_2=-13,\ x_1=72$

(3) $$\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 3 & 1\\ 2 & 3 & 1 & 1\\ 1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-5\\6\\4\\1\end{bmatrix}$$

Ans: $x_4=-1,\ x_3=2,\ x_2=1,\ x_1=0$

(4) $$\begin{bmatrix}3 & 2 & -1 & -4\\ 1 & -1 & 3 & -1\\ 2 & 1 & -3 & 0\\ 0 & -1 & 8 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}10\\-4\\16\\-3\end{bmatrix}$$

Ans: inconsistent. We cannot determine x_4 from the last eqn.

(5) $$\begin{bmatrix}2 & -1 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}$$

Ans: $x_5=x_4=x_3=x_2=x_1=1$

→ We can apply Gaussian elimination method to a system of eqns of any order. However, what happens if any one of the diagonal elements i.e, the pivots in the triangularization process vanishes. Then the method will fail. In such situations we modify the Gaussian elimination method and this procedure is called pivoting.

→ In the elimination process, if any one of the pivot elements $a_{11}^{1}, a_{22}^{2}, \ldots, a_{nn}^{n}$ vanishes or becomes very small compared to other elements in that row, then we attempt to rearrange the remaining rows so as to obtain a non-vanishing pivot or to avoid the multiplication by a large number. This strategy is called pivoting.

The pivoting is of the following two types:

(1) Partial pivoting: In the first stage of elimination, the first column is searched for the largest

element in magnitude and this largest element is then brought at the position of the first pivote by interchanging the first row with the row having the largest element in magnitude in the first column. For the second stage of elimination, the second column is searched for the largest element in magnitude among the (n−1) elements leaving the first element and then this largest element in magnitude is brought at the position of the second pivot by interchanging the second row with row having the largest element in the second column. This searching and interchanging of rows is repeated in all the (n−1) stages of the elimination.

Complete pivoting:

We search the matrix A for the largest element in magnitude and bring it as the first pivot. This requires not only an interchanging of eqns but also an interchange of the position of the variables.

Complete pivoting is much more complicated and is not often used.

→ Solve the system of eqns

$x_1 + x_2 + x_3 = 6$

$3x_1 + 3x_2 + 4x_3 = 20$

$2x_1 + x_2 + 3x_3 = 13$ Using Gauss elimination method with partial pivoting

Solⁿ. Now let us try first to solve the system

Without pivoting

we have $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

In the above matrix the second pivot has the value zero and the elimination procedure cannot be continued further unless, pivoting is used.

Now let us use the partial pivoting.

In the first column 3 is the largest element. Interchanging the rows 1st & 2nd, we get

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & -1 & 1/3 & -1/3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{1}{3}R_1 \\ R_3 \to R_3 - \frac{2}{3}R_1 \end{array}$$

In the second column, 1 is the largest element in magnitude leaving the first element.

Interchange the second and third rows, we have

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & -1 & 1/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \end{array}\right]$$

Clearly the resultant matrix is in triangular form and no further elimination is required.

Using the back substitution method, we obtain the solution $x_3 = 2,\ x_2 = 1,\ x_1 = 3$.

→ Solve the system of eqns
$0.0003x_1 + 1.566x_2 = 1.569$
$0.3454x_1 - 0.436x_2 = 3.018$
Using Gauss elimination method with and without pivoting.

Solⁿ: without pivoting

we have
$$[A|B] = \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -0.436 & 3.018 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0 & -1802.0 & -1803.0 \end{array}\right]$$

Clearly which is in triangular form and no further elimination is required.

Using back substitution method we obtain the solution $x_2 = 1.001$, $x_1 = 3.333$ — which is highly inaccurate compared to the exact solution.

with pivoting:
we interchange the 1st and 2nd rows
we get
$$[A|B] = \left[\begin{array}{cc|c} 0.3454 & -0.436 & 3.018 \\ 0.0003 & 1.566 & 1.569 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 0.345 & -0.436 & 3.018 \\ 0 & 1.566 & 1.566 \end{array}\right]$$

Clearly which is in triangular form and no further elimination is required.
Using back substitution method we obtain the solution $x_2 = 1$ & $x_1 = 10$
which is the exact solution.

HW → Solve the system of eqns

$x+y+z=7$
$3x+3y+4z=24$
$2x+y+3z=16$ by Gaussian elimination method with partial pivoting

Ans: $z=3,\ y=1,\ x=3$

→ Solve the Gaussian elimination method with partial pivoting, the following system of eqns

$0x_1+4x_2+2x_3+8x_4=24$
$4x_1+10x_2+5x_3+4x_4=32$
$4x_1+5x_2+6.5x_3+2x_4=26$
$9x_1+4x_2+4x_3+0x_4=21$

Ans: $x_1=1,\ x_2=1,\ x_3=2,\ x_4=2$

Gauss-Jordan Elimination method:

This method is a variation of the Gauss elimination method.

In the Gauss elimination method, using elementary row operations, we transform the matrix A to an upper triangular matrix U and obtain the solution by back substitution method.

In Gauss-Jordan elimination method not only the elements below the diagonal but also the elements above the diagonal of A are made zero at the same time.

In otherwords, we transform the matrix A to a diagonal matrix D. This diagonal matrix may then be reduced to an identity matrix by dividing each row

by its pivot element.

Alternatively, the diagonal elements can also be made unity at the same time when the reduction is performed.

This transforms the coefficient matrix into an identity matrix, on completion of the Gauss-Jordan method, we have

$$[A|B] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|d]$$

The solution is given by

$x_i = d_i$, $i = 1, 2, \ldots n$.

— pivoting can be used to make the pivot non-zero or to make it the largest element in magnitude in that column as discussed earlier.

— Generally the Gauss-Jordan elimination method require more number of operations compared to the Gaussian elimination method. We therefore, do not use this method for solving system of eqns but is very commonly used for finding the inverse matrix.

This is done by augmenting the matrix A by the identity matrix I of the order same as that of A. Using elementary row operations on the augmented matrix $[A|I]$ we reduce the matrix A to the form 'I' and in the process the matrix I is transformed to A^{-1}.

i.e., $[A|I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|A^{-1}]$

→ Solve the system of eqns

$x_1+x_2+x_3=1$
$4x_1+3x_2-x_3=6$
$3x_1+5x_2+3x_3=4$ by using the Gauss-Jordan method with pivoting.

Soln: we have

$$[A|B]=\left[\begin{array}{ccc|c}1&1&1&1\\4&3&-1&6\\3&5&3&4\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}4&3&-1&6\\1&1&1&1\\3&5&3&4\end{array}\right]$$ (Interchanging first & second row)

$$\sim\left[\begin{array}{ccc|c}4&3&-1&6\\0&\frac{1}{4}&\frac{5}{4}&-\frac{1}{2}\\0&\frac{11}{4}&\frac{15}{4}&-\frac{1}{2}\end{array}\right]$$ $R_2\to R_2-\frac{1}{4}R_1$, $R_3\to R_3-\frac{3}{4}R_1$

$$\sim\left[\begin{array}{ccc|c}4&3&-1&6\\0&\frac{11}{4}&\frac{15}{4}&-\frac{1}{2}\\0&\frac{1}{4}&\frac{5}{4}&-\frac{1}{2}\end{array}\right]$$ (Interchanging 2nd & 3rd row)

$$\sim\left[\begin{array}{ccc|c}4&0&-\frac{56}{11}&\frac{72}{11}\\0&\frac{11}{4}&\frac{15}{4}&-\frac{1}{2}\\0&0&\frac{10}{11}&-\frac{5}{11}\end{array}\right]$$ $R_3\to R_3-\frac{1}{11}R_2$, $R_1\to R_1-\frac{12}{11}R_2$

$$\sim\left[\begin{array}{ccc|c}4&0&0&4\\0&\frac{11}{4}&0&\frac{11}{8}\\0&0&\frac{10}{11}&\frac{5}{11}\end{array}\right]$$ $R_1\to R_1+\frac{56}{10}R_3$, $R_2\to R_2-\frac{33}{8}R_3$

$$\sim\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1/2\\0&0&1&-1/2\end{array}\right]$$ $R_1\to\frac{R_1}{4}$, $R_2\to\frac{4}{11}R_2$, $R_3\to\frac{11}{10}R_3$

which is the desired form.

$\therefore x_1=1;\ x_2=1/2;\ x_3=-1/2$.

→ Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$ using Gauss - Jordan method.

Sol:

Using the augmented matrix $[A|I]$,

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to \frac{1}{3}R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{7}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{7}{11} & [illegible] & -3/11 & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right] \quad R_2 \to \frac{-3}{11}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5/11 & 3/11 & 1/11 & 0 \\ 0 & 1 & 7/11 & 2/11 & -3/11 & 0 \\ 0 & 0 & 20/11 & 1/11 & -7/11 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - \frac{1}{3}R_2 \\ R_3 \to R_3 + \frac{7}{3}R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -5/11 & 3/11 & 1/11 & 0 \\ 0 & 1 & 7/11 & 2/11 & -3/11 & 0 \\ 0 & 0 & 1 & 1/20 & -7/20 & 11/20 \end{array}\right] \quad R_3 \to \frac{11}{20}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 1 & 0 & 3/20 & -1/20 & -7/20 \\ 0 & 0 & 1 & 1/20 & -7/20 & 11/20 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - \frac{5}{11}R_3 \\ R_2 \to R_2 - \frac{7}{11}R_3 \end{array}$$

$$= [I \mid A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 1/4 & 1/4 & -1/4 \\ 3/20 & -1/20 & -7/20 \\ 1/20 & -7/20 & 11/20 \end{bmatrix}$$

→ Find the inverse of the coefficient matrix of the system

$x_1+x_2+x_3=1$
$4x_1+3x_2-x_3=6$
$3x_1+5x_2+3x_3=4$ by the Gauss Jordan method with partial pivoting and hence solve the system.

Sol^n: we have

$$\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\6\\4\end{bmatrix}$$

Using the augmented matrix $[A|I]$, we obtain

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\4&3&-1&0&1&0\\3&5&3&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}4&3&-1&0&1&0\\1&1&1&1&0&0\\3&5&3&0&0&1\end{array}\right]\quad R_1\leftrightarrow R_2$$

$$\sim\left[\begin{array}{ccc|ccc}1&3/4&-1/4&0&\frac{1}{4}&0\\1&1&1&1&0&0\\3&5&3&0&0&1\end{array}\right]\quad R_1\to\frac{1}{4}R_1$$

$$\sim\left[\begin{array}{ccc|ccc}1&3/4&-1/4&0&1/4&0\\0&1/4&5/4&1&-1/4&0\\1&11/4&15/4&0&-3/4&1\end{array}\right]\quad \begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3-3R_1\end{array}$$

continuing in this way, we get

$$\sim\left[\begin{array}{ccc|ccc}1&0&0&7/5&1/5&-2/5\\0&1&0&-3/2&0&1/2\\0&0&1&11/10&-1/5&-1/10\end{array}\right]=[I|A^{-1}]$$

∴ solution of the system is

$$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=A^{-1}B=\begin{bmatrix}7/5&-1/5&-2/5\\-3/2&0&1/2\\11/10&-1/5&-1/10\end{bmatrix}\begin{bmatrix}1\\6\\4\end{bmatrix}=\begin{bmatrix}1\\1/2\\-1/2\end{bmatrix}$$

→ Find the inverse of the following matrices by using Gauss-Jordan method.

(1) $A=\begin{bmatrix}2&0&0&0\\1&1/2&0&0\\2&0&-3&0\\1&-1/2&-17&\frac{55}{3}\end{bmatrix}$ (2) $A=\begin{bmatrix}1&3/2&2&1/2\\0&1&-4&1\\0&0&1&2/3\\0&0&0&1\end{bmatrix}$ (3) $A=\begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}$

(4) $A=\begin{bmatrix}1&1&2\\1&2&4\\2&4&7\end{bmatrix}$

→ Using Gauss ~~elimination~~ Jordan method, find the inverse of the matrix $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$

3.8.1 Gaussian Elimination Method

In this method, if A is a given matrix, for which we have to find the inverse; at first, we place an identity matrix, whose order is same as that of A, adjacent to A which we call an *augmented* matrix. Then the inverse of A is computed in two stages. In the first stage, A is converted into an upper triangular form, using Gaussian elimination method as discussed in Section 3.2. In the second stage, the above upper triangular matrix is reduced to an identity matrix by row transformations. All these operations are also performed on the adjacently placed identity matrix. Finally, when A is transformed into an identity matrix, the adjacent matrix gives the inverse of A. In order to increase the accuracy of the result, it is essential to employ partial pivoting. To understand the sequence of the steps involved, we consider an example.

Example 3.9 Use the Gaussian elimination method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Solution At first, we place an identity matrix of the same order adjacent to the given matrix. Thus, the augmented matrix can be written as

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \qquad (1)$$

Stage I (Reduction to upper triangular form): Let R_1, R_2 and R_3 denote the first, second and third rows of a matrix. In the first column of Eq. (1), 4 is the largest element, thus interchanging R_1 and R_2 to bring the pivot element 4 to the place of a_{11}, we have the augmented matrix in the form

$$\left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \qquad (2)$$

Divide R_1 by 4 to get

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \qquad (3)$$

Perform $R_2 - R_1 \longrightarrow R_2$, which gives

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{4}$$

Perform $R_3 - 3R_1 \longrightarrow R_3$ in Eq. (4), which yields

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \end{array}\right] \tag{5}$$

Now, looking at the second column for the pivot, the max (1/4, 11/4) is 11/4. Therefore, we interchange R_2 and R_3 in Eq. (5) and get

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right] \tag{6}$$

Now, divide R_2 by the pivot $a_{22} = 11/4$, and obtain

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right] \tag{7}$$

Performing $R_3 - (1/4)R_2 \longrightarrow R_3$ in (7) yields

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{11} & -\frac{1}{11} \end{array}\right] \tag{8}$$

Finally, we divide R_3 by (10/11), thus getting an upper triangular form

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \qquad (9)$$

Stage II (Reduction to an identity matrix): Multiply R_3 by $-1/4$ and $15/11$ respectively and subtract it from R_1 and R_2 of Eq. (9), we get

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \qquad (10)$$

Finally, performing $R_1 - (3/4)\ R_2 \longrightarrow R_1$ in Eq. (10), we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right]$$

Thus, we have

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix} \qquad (11)$$

We can easily cheque $[A]\ [A^{-1}] = [I]$.

3.8.2 Gauss–Jordan Method

This method is similar to Gaussian elimination method, with the essential difference that the stage *I* of reducing the given matrix to an upper triangular form is not needed. However, the given matrix can be directly reduced to an identity matrix using elementary row transformations. This technique is illustrated in the following example.

Example 3.10 Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

by Gauss–Jordan method.

Solution. Let R_1, R_2 and R_3 denote the first, second and third rows of a matrix. We place an identity matrix adjacent to the given matrix as a first step and the resulting augmented matrix is given by

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

Performing $R_2 - 4R_1 \longrightarrow R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

Now, performing $R_3 - 3R_1 \longrightarrow R_3$, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \tag{3}$$

Carrying out further operations $R_2 + R_1 \longrightarrow R_1$ and $R_3 + 2R_2 \longrightarrow R_3$, we arrive at

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \tag{4}$$

Now, dividing the third row by -10, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \tag{5}$$

proceeding in this way we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] = [I \mid A^{-1}]$$

Indirect methods:-

Iteration Method:

Direct methods provide the exact solution in a finite number of steps provided exact arithmetic is used and there is no round-off error. Also direct methods are generally used when the matrix A is having few zero elements and the order of the matrix is not very large say $n<50$.

Iterative methods, on the other hand, start with an initial approximation and by applying a suitably chosen algorithm, lead to successively better approximations. Even if the process converges, it gives only an approximate solution. These methods are generally used when the matrix A is sparse (many elements are zero) and the order of the matrix 'A' is very large say $n>50$. Sparse matrices have very few non-zero elements. In most cases these non-zero elements lie on or near the main diagonal giving rise to trigonal, or five diagonal matrix systems.

It may be noted that there are no fixed rules to decide when to use direct methods and when to use iterative methods.

However, when the coefficient matrix is sparse or large, the use of iterative methods is ideally suited to find the solution which take advantage of the sparse nature of the matrix involved.

The General Iteration Method

We start with some initial approximate solution vector $x^{(0)}$ and generate a sequence of approximations $\{x^{(k)}\}$ which converge to the exact solution vector x as $k \to \infty$. If the method is convergent, each iteration produces a better approximation to the exact solution. We repeat the iterations till the required accuracy is obtained.

Therefore, in an iterative method the amount of computation depends on the desired accuracy whereas in direct methods the amount of computation is fixed. The number of iterations needed to obtain the desired accuracy also depends on the initial approximation, closer the initial approximation to the exact solution, faster will be the convergence.

→ Now consider the system of eqns

$$AX = B \quad \text{--- } (1)$$

where A is $n \times n$ non-singular matrix.

$$\Rightarrow \left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \text{--- } (2)$$

We assume the diagonal coefficients $a_{ii} \neq 0$ $(i = 1, 2, \cdots n)$

If some $a_{ii} = 0$ then we arrange the eqns, so that this condition holds

Now we rewrite the system ② as

$$x_1 = \frac{-1}{a_{11}}(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n) + \frac{b_1}{a_{11}}$$

$$x_2 = \frac{-1}{a_{22}}(a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n) + \frac{b_2}{a_{22}}$$

$$\cdots \cdots \cdots \cdots$$

$$x_n = \frac{-1}{a_{nn}}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1}) + \frac{b_n}{a_{nn}} \quad \text{— ③}$$

In matrix form, system ③ can be written as

$X = HX + C$

$$\text{where } H = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots & -\frac{a_{2n}}{a_{22}} \\ \vdots & & & & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & -\frac{a_{nn-1}}{a_{nn}} & 0 \end{bmatrix} \quad \text{— ④}$$

and the elements of C are $c_i = \frac{b_i}{a_{ii}}$ $(i = 1, 2, \ldots n)$

To solve ③, we make an ~~initial~~ guess $x^{(0)}$ of ~~the~~ solution vector and substitute into the RHS of eqn ③. The solution of ~~equation~~ ③ will then yield a vector $x^{(1)}$, which hopefully is a better approximation to the solution ~~than~~ $x^{(0)}$. We then substitute $x^{(1)}$ into the RHS of eqn ③ and get another approximation $x^{(2)}$. We continue in this manner until the successive iterations $x^{(k)}$ have converged to the required number of significant figures.

In general we can write the iteration method for solving the linear system

of eqns (1) in the form $x^{(k+1)} = Hx^{(k)} + c$, ——(5)
$k = 0, 1, 2 \ldots$

where $x^{(k)}$ and $x^{(k+1)}$ are the approximations for x at the k^{th} and $(k+1)^{th}$ iterations respectively.

H is called the iteration matrix and depends on A and c is a column vector and depends on both A and B.

The matrix H is generally a constant matrix when the method (5) is cgt, then

$$\lim_{k\to\infty} x^{(k)} = \lim_{k\to\infty} x^{(k+1)} = x$$

and we obtain from eqn (5),

$$\boxed{x = Hx + c} \longleftarrow (6)$$

If we define the error vector at the k^{th} iteration as $\epsilon^{(k)} = x^{(k)} - x$ ——(7)

then subtracting eqn (6) from eqn (5) (i.e (5)−(6)) we obtain

$$x^{(k+1)} - x = H\left[x^{(k)} - x\right]$$

$$\Rightarrow x^{(k+1)} - x = H\epsilon^{(k)}$$

$$\Rightarrow \epsilon^{(k+1)} = H\epsilon^{(k)} \quad \left(\because \epsilon^{(k)} = x^{(k)} - x\right)$$
——(8)

(8) ≡ $\epsilon^{(k)} = H\epsilon^{(k-1)}$
$= H\left(H\epsilon^{(k-2)}\right)$
$= H^2\epsilon^{(k-2)}$
$= H^2\left(H\epsilon^{(k-3)}\right)$
$= H^3\epsilon^{(k-3)} = \cdots\cdots = H^k\epsilon^{(0)}$.

where $\epsilon^{(0)}$ is the error in the initial approximate vector. Thus, for the convergence of the iterative method, we must have $\lim_{k\to\infty} \epsilon^{(k)} = 0$ independent of $\epsilon^{(0)}$.

Gauss-Seidel Iteration method:

Consider the system of eqns (2) written in form (3) for this system of eqns, we define the Gauss-Seidel method as:

$$x_1^{(k+1)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1\right)$$

$$x_2^{(k+1)} = -\frac{1}{a_{22}}\left(a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)} - b_2\right)$$

$$\vdots$$

$$x_n^{(k+1)} = -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \cdots + a_{n,n-1}x_{n-1}^{(k+1)} - b_n\right)$$

(I)

(or)

$$x_i^{(k+1)} = -\frac{1}{a_{ii}}\left[\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} - b_i\right]$$

$i = 1, 2, 3, \ldots n$

Note that, in the first eqn of system (I), we substitute the initial approximation $\left(x_2^{(0)}, x_3^{(0)}, \ldots x_n^{(0)}\right)$ on RHS.

In the second eqn; we substitute $\left(x_1^{(1)}, x_3^{(0)}, \ldots x_n^{(0)}\right)$ on RHS.

In third eqn, we substitute $\left(x_1^{(1)}, x_2^{(1)}, x_4^{(0)}, \ldots x_n^{(0)}\right)$.

We continue in this manner until all the components have been improved. At the end of this first iteration, we will have an improved vector $\left(x_1^{(1)}, x_2^{(1)}, \ldots x_n^{(1)}\right)$.

The entire process is then repeated. In other words the method uses an improved component as soon as it becomes available. It is

for this reason the method is also called the method of successive displacements.

Q. → perform four iterations (rounded to four decimal places) using the Gauss-seidel method for solving the system of eqns

$$\begin{bmatrix} -8 & 1 & 1 \\ 1 & -5 & 1 \\ 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \\ 7 \end{bmatrix} \quad \text{with } x^{(0)} = 0$$

The exact solution is $X = (-1\ -4\ -3)^T$

Soln: Given system is

$$\left.\begin{aligned} -8x_1 + x_2 + x_3 &= 1 \\ x_1 - 5x_2 + x_3 &= 16 \\ x_1 + x_2 - 4x_3 &= 7 \end{aligned}\right\} \text{—— (i)}$$

$$\Rightarrow \left.\begin{aligned} x_1 &= -\frac{1}{8}(1 - x_2 - x_3) \\ x_2 &= -\frac{1}{5}(16 - x_1 - x_3) \\ x_3 &= -\frac{1}{4}(7 - x_1 - x_2) \end{aligned}\right\} \text{—— (ii)}$$

By the Gauss-seidel method, system (ii) can be written as

$$x_1^{(k+1)} = -\frac{1}{8}\left(1 - x_2^{(k)} - x_3^{(k)}\right)$$

$$x_2^{(k+1)} = -\frac{1}{5}\left(16 - x_1^{(k+1)} - x_3^{(k)}\right)$$

$$x_3^{(k+1)} = -\frac{1}{4}\left(7 - x_1^{(k+1)} - x_2^{(k+1)}\right)$$

where $k = 0, 1, 2, \ldots$

Now taking $x^{(0)} = 0$, we obtain the following iterations

$k = 0$ –

$$x_1^{(1)} = -\frac{1}{8}\left(1 - x_2^{(0)} - x_3^{(0)}\right) = -\frac{1}{8}(1 - 0 - 0)$$

$$= -\frac{1}{8} = -0.125$$

$$x_2^{(1)} = -\frac{1}{5}\left[16 - x_1^{(1)} - x_3^{(0)}\right]$$
$$= -\frac{1}{5}\left[16 + 0.125 - 0\right]$$
$$= -3.225$$

$$x_3^{(1)} = -\frac{1}{4}\left[7 - x_1^{(1)} - x_2^{(1)}\right]$$
$$= -\frac{1}{4}\left[7 + 0.125 + 3.225\right]$$
$$= -2.5875$$

K=1:

$$x_1^{(2)} = -\frac{1}{8}\left[1 - x_2^{(1)} - x_3^{(1)}\right]$$
$$= -\frac{1}{8}\left[1 + 3.225 + 2.5875\right]$$
$$= -0.8516$$

$$x_2^{(2)} = -\frac{1}{5}\left[16 - x_1^{(2)} - x_3^{(1)}\right]$$
$$= -\frac{1}{5}\left[16 + 0.8516 + 2.5875\right]$$
$$= -3.8878$$

$$x_3^{(2)} = -\frac{1}{4}\left[7 - x_1^{(2)} - x_2^{(2)}\right]$$
$$= -\frac{1}{4}\left[7 + 0.8516 + 3.8878\right]$$
$$= -2.9349$$

K=2:

$$x_1^{(3)} = -\frac{1}{8}\left[1 - x_2^{(2)} - x_3^{(2)}\right]$$
$$= -\frac{1}{8}\left[1 + 3.8878 + 2.9349\right]$$
$$= -0.9778$$

$$x_2^{(3)} = -\frac{1}{5}\left[16 - x_1^{(3)} - x_3^{(2)}\right]$$
$$= -\frac{1}{5}\left[16 + 0.9778 + 2.9349\right]$$
$$= -3.9825$$

$$x_3^{(3)} = -\frac{1}{4}\left[7 - x_1^{(3)} - x_2^{(3)}\right]$$
$$= -\frac{1}{4}\left[7 + 0.9778 + 3.9825\right]$$
$$= -2.9901$$

K=3 :

$$x_1^{(4)} = -\frac{1}{8}[1+3.9825+2.9901]$$
$$= -0.9966$$

$$x_2^{(4)} = -\frac{1}{5}[16+\bar{0}.9966+2.9901]$$
$$= -3.9973$$

$$x_3^{(4)} = -\frac{1}{4}[7+0.9966+3.9973]$$
$$= -2.9985$$

which is a good approximation to the exact solution $x = (-1\ -4\ -3)^T$ with maximum error 0.0034

→ Solve the following eqns

$$2x_1 - x_2 + 0x_3 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0x_1 - x_2 + 2x_3 = 1$$

using Gauss-seidel method of iteration and perform three iterations.

Sol: The Given system of eqns can be written as

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(7+x_2) \\ x_2 &= \frac{1}{2}(1+x_1+x_3) \\ x_3 &= \frac{1}{2}(1+x_2) \end{aligned}\right] \longrightarrow (1)$$

By the Gauss-seidel method, system (1) can be written as

$$x_1^{(k+1)} = \frac{1}{2}\left(7+x_2^{(k)}\right)$$
$$x_2^{(k+1)} = \frac{1}{2}\left(1+x_1^{(k+1)}+x_3^{(k)}\right)$$
$$x_3^{(k+1)} = \frac{1}{2}\left(1+x_2^{(k+1)}\right)$$

where $k=0,1,2,\ldots$

Now taking $x^{(0)} = 0$, we obtain the following iterations

k = 0 : $x_1^{(1)} = \frac{1}{2}(7+0) = \frac{7}{2} = 3.5$

$x_2^{(1)} = \frac{1}{2}\left(1 + x_1^{(1)} + x_3^{(0)}\right)$

$= \frac{1}{2}(1+3.5+0)$

$= \frac{4.5}{2} = 2.25$

$x_3^{(1)} = \frac{1}{2}\left(1 + x_2^{(1)}\right)$

$= \frac{1}{2}(1+2.25) = \frac{1}{2}(3.25)$

$= 1.625$

k = 1 : $x_1^{(2)} = \frac{1}{2}\left(7 + x_2^{(1)}\right)$

$= \frac{1}{2}(7+2.25) = \frac{9.25}{2} = 4.625$

$x_2^{(2)} = \frac{1}{2}\left(1 + x_1^{(2)} + x_3^{(1)}\right)$

$= \frac{1}{2}(1+4.625+1.625)$

$= \frac{1}{2}(7.250) = 3.625$

$x_3^{(2)} = \frac{1}{2}\left(1 + x_2^{(2)}\right)$

$= \frac{1}{2}(1+3.625) = \frac{4.625}{2} = 2.3125$

k = 2 : $x_1^{(3)} = \frac{1}{2}\left(7 + x_2^{(2)}\right)$

$= \frac{1}{2}(7+3.625) = \frac{10.625}{2} = 5.3125$

$x_2^{(3)} = \frac{1}{2}\left(1 + x_1^{(3)} + x_3^{(2)}\right)$

$= \frac{1}{2}(1+5.3125+2.3125)$

$= \frac{8.625}{2} = 4.3125$

$x_3^{(3)} = \frac{1}{2}\left(1 + x_2^{(3)}\right)$

$= \frac{1}{2}(1+4.3125) = 2.6563.$

→ Use the Gauss-Seidel method for solving the following system of eqns

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

with $x^{(0)} = [0.5 \quad 0.5 \quad 0.5 \quad 0.5]^T$.

Compare the results with the exact solution is $X = [1 \ 1 \ 1 \ 1]^T$.

2001, 2011 → Using the Gauss-Seidel method and starting solution $x_1 = x_2 = x_3 = 0$, determine the solution of the following system of eqns in two iterations

$$10x_1 - x_2 - x_3 = 8$$
$$x_1 + 10x_2 + x_3 = 12$$
$$x_1 - x_2 + 10x_3 = 10$$

Compare the approximate solution with the exact solution

2009, 15M → Using Gauss-Seidel iterative method, find the solution of the following system:

$$4x - y + 8z = 26$$
$$5x + 2y - z = 6$$
$$x - 10y + 2z = -13.$$ up to three iterations

→ Find the solution of the following system of eqns

$$x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 = \frac{1}{2}$$
$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$
$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$
$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$ using Gauss Seidel and perform the first five iterations

→ Solve the system eqns

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 2z = 25$$ by Gauss-Seidel iterative method and perform the first three iterations